Bianchi

Note book

Function

Theory.

2

References to Forsyth by articles, thus: [ ]

I

Theory of Functions.

Cornell University, Summer 1904.

John Irwin Hutchinson.

1) Interpretation (geometrical) of the complex variable. $z = x + iy = r\varepsilon^{i\theta}$.

Addition, subtraction, multiplication, interpreted geometrically by expressing in terms of the coordinates (x, y), or (r, θ)) [1–3]

Let $w = f(z)$ and suppose that if $z = x + iy$ w can be represented in the form $w = u + iv$. Then $\frac{\delta w}{\delta x} = \frac{dw}{dz}$; $\frac{\delta w}{\delta y} = i\frac{dw}{dz}$ whence

2) (1) $\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y}$; $\frac{\delta u}{\delta y} = -\frac{\delta v}{\delta x}$

3) [This is the condition that any expression of the form $u + iv$ can be expressed as $f(z)$.]

These are also the conditions that $\frac{dw}{dz}$ shall be unique [That is that there shall be one slope or tangent in the w-plane regardless of the method of approach to a point or the slope $\frac{dy}{dx}$ in the z plane]. For

$$\frac{dw}{dz} = \frac{du + i\,dv}{dx + i\,dy} = \frac{\frac{\delta u}{\delta x} + \frac{\delta u}{\delta y}y' + i\left(\frac{\delta v}{\delta x} + \frac{\delta v}{\delta y}y'\right)}{1 + iy'} \quad \left\{y' = \frac{dy}{dx}\right.$$

To be independent of y', it is necessary for the numerator to be divisible by $1 + iy'$

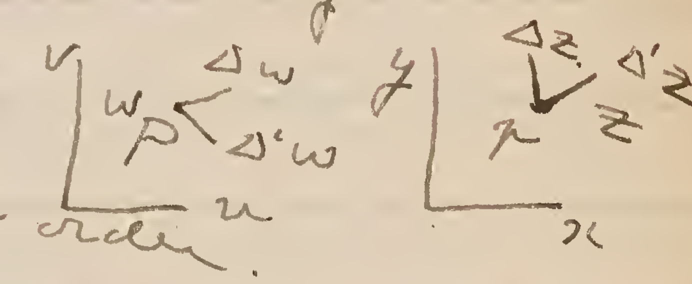
whence $\frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x} = \left(\frac{1}{i}\frac{\delta u}{\delta y} + \frac{\delta v}{\delta y}\right)$ [Equating real and imag. parts]

and hence the above conditions.

(Picard II. p. 2.)

4) (2) From these relations follow $\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = 0$, $\frac{\delta^2 v}{\delta x^2} + \frac{\delta^2 v}{\delta y^2} = 0$.
(or u & v satisfy the same Dif. Eq.)

Let Δw, $\Delta' w$ correspond to Δz, $\Delta' z$. Then by Maclaurin's formula.



$$\Delta w = \frac{dw}{dz}\Delta z + \text{infinitesimals of higher order.}$$
$$\Delta' w = \frac{dw}{dz}\Delta' z + \quad " \quad " \quad " \quad "$$

Neglecting infinitesimals of higher order, we have

$$\frac{\Delta w}{\Delta' w} = \frac{\Delta z}{\Delta' z}$$

5) whence, any two infinitesimals lines radiating from a point in the z-plane correspond to similar lines in the w-plane, which are proportional to the two lines to which they correspond, and hence, An infinitesimal triangle in the z-plane has corresponding to it a similar triangle in the w-plane. This geometrical relation is called conformal. Example

6) $w = \frac{1}{z}$ (Transformation by reciprocal radii)
The w plane is related to the z plane in such a way that
circles correspond to circles. — End lecture I June 8 1904 9 a.m.

7) Ex. $w = \log z$. P 110

8) Ex. Show that the relation $2z = (a-b)w + \frac{a+b}{w}$ is such that to circles in the w plane concentric with the origin, correspond confocal ellipses in the z plane, and lines radiating from the origin correspond to confocal hyperbolas.
[§ 257, Ex 1.]

9) If either the real or the imaginary part of a function of a complex variable be given, the other part is determined except for an

arbitrary additive constant. For example
Let u be given. Then

$$dv = \frac{\delta v}{\delta x}dx + \frac{\delta v}{\delta y}dy = -\frac{\delta u}{\delta y}dx + \frac{\delta u}{\delta x}dy$$

and hence $v = \int\left(-\frac{\delta u}{\delta y}dx + \frac{\delta u}{\delta x}dy\right) + C$ [§11]

(Integral of perfect differential eq. from 4) p. 4.)

10) If a function have only one value for each given value of the variable it is called _uniform_

11) If the function have more than one value, or if its value is modified by changing the path by which the variable reaches ~~that~~ given value, it is _multiform_. e.g. Log z.

12) _Branch_ of a _function_

13) The _branch_ points.

14) A function which is continuous and uniform over any given part of the z-plane is called _holomorphic_ over that part of the plane.

15) A _root_ (or zero) of a function.

16) An _Infinite_ of a function {(1) poles / (2) not poles
In case (1) the reciprocal of the function is holomorphic. in (2) it is not.

17) The infinites are called _singularities_

18) A function which is uniform and continuous except at poles is called _meromorphic_. exs. $\frac{F(z)}{\phi(z)}$ (F & ϕ polynomials) $\log z$, e^z

cf. Forsythe

19) _Integration_ of _Uniform_ _Functions_

$$\int_a^z f(z)\,dz = \lim\left[(z_1 - a)f(a) + (z_2 - z_1)f(z_1) + \cdots + (z - z_n)f(z_n)\right]$$

20) The Integral may also be defined as the sum of two curvilinear Integrals, thus.

$$f(z) = u + iv.$$

$$\int_a^z f(z)\,dz = \int(u\,dx - v\,dy) + i\int(v\,dx + u\,dy)$$

These are integrals of total differentials (eqs top page 4), and hence are independent of the path of integration.

21) Properties. 1. $\int_a^z = \int_a^\xi + \int_\xi^z$

22) 2. $\int_a^z = -\int_z^a$

End of Lecture II.

23) 3. The integral of a sum of a finite number of terms, equals the sum of the integrals of the separate terms, the path of integration being the same for all.

24) 4. If $f(z)$ is finite and continuous along any line between a and z, $I = \int_a^z f(z)\,dz$ is finite.

See top of page 14 for (24a) Proof of above.

25) Lemma. If p and q be any two functions of x, y which are uniform, finite and continuous over a given region D. of the xy plane then

$$3 \quad \iint_D \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) dx\,dy = \int_S (p\,dx + q\,dy)$$

the second integral being taken over the boundary of the region in a pos. direction.

[16]

26) If $f(z)$ is holomorphic over any region of the z-plane then the integral $f(z)\,dz$ taken around the complete boundary of that region

(positively defined) is zero.

For let $p = f(z)$, $q = ip$

$\therefore \frac{\partial p}{\partial x} = \frac{1}{i}\frac{\partial p}{\partial y}$

& $\frac{\partial q}{\partial x} = i\frac{dp}{\partial x} = \frac{\partial p}{\partial y}$

$$\therefore \iint_{(S)} \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) dx\,dy = 0$$

$$\therefore \int (p\,dx + q\,dy) = 0$$

But $\int (p\,dx + q\,dy) = \int f(z)\,dx + i f(z)\,dy$

$= f(z)\,dz = 0$ [17]

27) If $f(z)$ is holomorphic over a portion of the plane bounded by two simple curves (one within the other) the integral $\int f(z)\,dz$ takes the same value when integrated around either curve in the same direction.

28) Cor. When the integral $f(z)dz$ is taken around the whole of any simple curve in the plane, no change is caused in its value by continuously deforming the curve into any other simple curve provided that the function is holomorphic over the part of the plane in which the deformation is made. [19]

Similarly for any number of curves.

29) Let $f(z)$ denote a function which is holomorphic over any region in the z plane and let a denote any point within that

region which is not a zero of $f(z)$, then $f(a) = \frac{1}{2\pi i}\int \frac{f(z)\,dz}{z-a}$, the integral being taken positively around the entire boundary of that region.

30) The theorem can be stated otherwise as follows: If $g(z)$ is a meromorphic function such that $(z-a)g(z)$ is holomorphic in a region enclosing a and $\lim_{z=a}(z-a)\,g(z)$ is finite and not zero.

then $\frac{1}{2\pi i}\int g(z)\,dz = \lim_{z=a}(z-a)\,g(z)$ the integral being taken around a curve in a region enclosing a. [p 28] End Lect 3

31) Differentiation of an integral with respect to a parameter

Let $f(z)$ be holomorphic within a given region. Let a be a point within this region which is ~~not~~ a zero of $f(z)$. Let $a+\delta a$ be any other point of the region in the vicinity of a. Then.

$$f(a+\delta a) = \frac{1}{2\pi i}\int_{(B)} \frac{f(z)}{z-a-\delta a}\,dz. \quad \text{[Integrated positively about boundary.]}$$

$$\text{Hence } f(a+\delta a) - f(a) = \frac{1}{2\pi i}\int_{(B)}\left[\frac{1}{z-a-\delta a} - \frac{1}{z-a}\right]dz\cdot f(z)$$

$$= \frac{1}{2\pi i}\int_{(B)}\left[\frac{\delta a}{(z-a)^2} + \frac{(\delta a)^2}{(z-a)^2(z-a-\delta a)}\right] f(z)\,dz.$$

Since $f(z)$ is a holomorphic function of z it has a unique derivative at each point of the region considered and hence

$$\lim_{\delta a=0}\left[\frac{f(a+\delta a)-f(a)}{\delta a}\right] = f'(a)$$

or $\dfrac{f(a+\delta a) - f(a)}{\delta a} = f'(a) + \sigma$ where $\lim\limits_{\delta a = 0} \sigma = 0$

Hence $f'(a) - \dfrac{1}{2\pi i}\displaystyle\int \frac{f(z)\,dz}{(z-a)^2} = -\sigma + \frac{\delta a}{2\pi i}\int \frac{f(z)\,dz}{(z-a)^2(z-a-\delta a)}$

δa being a constant so far as the integration relative to z is concerned.

Then $\left| f'(a) - \dfrac{1}{2\pi i}\displaystyle\int \frac{f(z)\,dz}{(z-a)^2} \right| = \left| -\sigma + \frac{\delta a}{2\pi i}\int \frac{f(z)\,dz}{(z-a)^2(z-a-\delta a)} \right|$

$$< |\sigma| + \frac{|\delta a|}{2\pi}\left| \int \frac{f(z)\,dz}{(z-a)^2(z-a-\delta a)} \right|$$

sum of moduli > modulus of sum.

Let M be the greatest modulus of $\dfrac{f(z)}{(z-a)(z-a-\delta a)}$ for points on the boundary

This is finite provided that a and $a+\delta a$ are not infinitesimally near the boundary.

[Since the integral is integrated around the boundary z is on the boundary as]

$\left(\dfrac{|a}{\delta a}\right)^z$ Then $\left| \displaystyle\int \frac{f(z)\,dz}{(z-a)^2[z-a-\delta a]} \right| < MS$

where S is the length of arc of boundary supposed finite. Hence

$$\left| f'(a) - \frac{1}{2\pi i}\int \frac{f(z)\,dz}{(z-a)^2} \right| < |\sigma| + \frac{|\delta a|}{2\pi} MS$$

When we take the limit the right member vanishes and hence

$$f'(a) = \frac{1}{2\pi i}\int \frac{f(z)\,dz}{(z-a)^2}$$ a process of differentiation with ref. to a.

32) The function $f'(z)$ is holomorphic within the given region, For by the same method as above it can be shown that

$|f'(a+\delta a) - f'(a)|$ can be made less than any assigned quantity.

Further $f'(a)$ is uniform in the given region, for it is the limit of $\frac{f(a+\delta a)-f(a)}{\delta a}$ and $f(a+da)$ and $f(a)$ are both uniform, Lastly $f'(a)$ is finite for it is the value of the integral $\frac{1}{2\pi i}\int \frac{f(z)\,dz}{(z-a)^2}$ in which the path of integration is finite and the subject of integration is finite at every point of the path. Hence $f'(a)$ is a holomorphic function of $\underline{a}$. We can accordingly repeat the preceding operations of $f'(a)$ and obtain $f''(a) = \frac{2!}{2\pi i}\int_{(B)} \frac{f(z)}{(z-a)^3}\,dz$ and by repetition

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_{(B)} \frac{f(z)\,dz}{(z-a)^{n+1}}$$

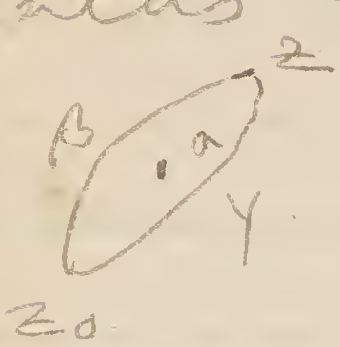
33) Hence:

When a function is holomorphic within a given region of the plane bounded by a simple curve of finite length it has an unlimited number of derivatives each of which is holomorphic within that region.

34) <u>Integral of a meromorphic function along an open path</u>:

If $f(z)$ be holomorphic in a given region then the meromorphic function $\frac{f(z)}{z-a}$ when integrated along two different paths

from z_0 to z will attain the same value if the point $\underline{a}$ is not enclosed by the two paths. If it is then



$$\int_{(z_0\beta z)} \frac{f(z)}{z-a}\,dz = \int_{z_0\,(\gamma)}^{z} \frac{f z}{z-a}\,dz + 2\pi i\, f(a)$$

$$\int_{z_0}^{z} + \int_{z\,\gamma}^{z_0} = 2\pi i\, f(a) = \int_{(z_0\beta z\gamma z_0)}$$

35) <u>Theorem</u>

If $f(z)$ be such that $\lim_{z=a} (z-a) f z = 0$ and if $f(z)$ has no infinites or branch points in the immediate vicinity of $\underline{a}$ the value of $\int f(z)\,dz$ taken around a small circle with its center at $\underline{a}$ tends toward zero when the circle diminishes in magnitude so as to ultimately have a vanishing radius, for let $z - a = r\varepsilon^{i\theta}$ $\frac{dz}{z} = i\,d\theta$.

$$\int f(z)\,dz = i\int_0^{2\pi} (z-a)\, f(z)\, d\theta.$$

Hence
$$\left|\int f(z)\,dz\right| = \left|\int_0^{2\pi} (z-a)\, f(z)\, dz\right|$$
$$< \int_0^{2\pi} \left|(z-a) f(z)\right| d\theta$$
$$< \int_0^{2\pi} M\, d\theta.$$
$$\leq 2\pi M$$ where M is the greatest value of $|(z-a) f(z)|$ on the circumference. Since $(z-a) f(z)$ tends toward zero as $|z-a|$ diminishes, it follows that $\lim M = 0$ and hence the theorem. <u>Remark</u> if the integral is extended over only part of the circumference the theorem is true under the same conditions (replace θ by θ_1 and 2π by θ_2)

36) <u>Cor.</u> If $\lim (z-a) f(z) = c$. ~~then~~
$$\lim_{r=0} \int f(z)\, dz = 2\pi i c \qquad r = |z-a|$$
the integration being positive around a circle of radius r and center a.

Ex. $\lim \int \frac{dz}{(a^2-z^2)^{1/2}}$ integrated around a very small circle center at $\underline{a}$ $(a \neq 0)$ is zero.

Ex $\lim \int \frac{dz}{(a-z)(z+a)^{1/2}} = \frac{\pi}{i}\left(\frac{2}{a}\right)^{1/2}$

37) <u>Theorem</u> If $z f(z)$ is finite for sufficiently large values of z and tends uniformly to zero as $\underline{z}$ increases indefinitely, then $\lim_{R=\infty} \int f(z)\, dz = 0$ the integration being around a circle of radius R, (origin as center) $R = |z|$

for
$$\int f(z)\, dz = i \int_0^{2\pi} z f(z)\, d\theta.$$
$$< 2\pi M \doteq 0 \text{ as } M \doteq 0$$

38) Cor. If $\lim_{R=\infty} z f(z) = c$ then
$$\lim_{R=\infty} \int f(z)\, dz = 2\pi i c$$

Ex $\lim_{R=\infty} \int \frac{dz}{\sqrt{1-z^n}} = \begin{matrix} 2\pi & n=2 \\ 0 & n>2 \end{matrix}$

39) Theorem. If all the infinities and the branch points of a function lie in a finite region of the z plane and the $\lim\limits_{z=\infty} z f z = 0$ the $\int f(z)dz$ is zero when integrated around any simple closed curve which includes all the points. For the curve can be deformed into the infinite circle without passing over any singularities of the function.

40) Cor. If under the same circumstances, $\lim\limits_{R=\infty} z f(z) = c$, then the integral is $2\pi i c$.

End Lect. 4.

Ex $\int \frac{dz}{\sqrt{a^2-z^2}}$ along a closed curve including $\underline{a}$ and $\underline{-a}$ is 2π: $R = |z|$

see bottom p 14

Ex. $\int \frac{dz}{\sqrt{(1-z^2)(c^2-z^2)}}$ along a closed curve including ± 1, $\pm c$ is zero.

Ex. $\int \frac{dz}{\sqrt{(z-\varepsilon_1)(z-\varepsilon_2)(z-\varepsilon_3)}}$ is not zero.

when integrated around a path including $\varepsilon_1, \varepsilon_2\, \varepsilon_3$ since $z = \infty$ is a branch point

but the limit of this as z increases indefinitely is zero.

Ex $\int \varepsilon^{-z^2} dz$ – (See Forsyth Art 25 p 38.)

 To follow sec. (2-4) p. 6.

24a) Since $I = \lim_{n=\infty} \sum_{r=0}^{n} (z_{r+1} - z_r) f(z_r)$

$$|I| < \lim \sum |(z_{r+1} - z_r)| \cdot |f(z_r)|$$

Let M be the upper limit of $|f(z_r)|$. Then

$$|I| < \lim M \sum |z_{r+1} - z_r|$$
$$< M S.$$

where S is the length of the ~~curve~~ path of integration

5. If $f(z) = \sum_{i=0}^{n} u_i + R$. where $|R|$ can be made as small as we please by sufficiently increasing n, then

$$\int_a^z f(z)\,dz = \sum_{i=0}^{\infty} \int_a^z u_i\,dz$$ within the region of convergence

For let $\Theta = \int_a^z R\,dz$

then $$\int_a^z f(z)\,dz - \sum_{i=0}^{n} \int_a^z u_i\,dz = \Theta$$

If R' denote the greatest value of $|R|$ for points in the path of integration

then by (4) $\Theta = \int_a^z R\,dz < R'S \doteq 0$ [§15]

[In the last two problems, p.13 the including of the points apparently plays no role. but provision must be made for this. otherwise the theorem used is not applicable. This is one of the conditions of it applicability]

Expansion of Functions in Series

When a function is holomorphic over a circle of center a it can be expanded into a series of positive integral powers of $(z-a)$ converging for all points within the circle

Let z be any point within the circle. Choose r so that $|z-a| = \rho < r < R$. where R is the radius of the given circle. Construct a new circle with center a and radius r. If t denote the value of the complex variable for any point on the circumference of this circle, then

$$f(z) = \frac{1}{2\pi i}\int \frac{f(t)\,dt}{t-z}$$ integrated positively around the circle

$$= \frac{1}{2\pi i}\int \frac{f(t)\,dt}{(t-a)\left[1-\frac{z-a}{t-a}\right]}, \quad (z-a < t-a)$$

now expand $\dfrac{1}{1-\frac{z-a}{t-a}}$ by division

we then obtain, after integrating,

$$f(z) = \frac{1}{2\pi i}\int \frac{f(t)\,dt}{t-a} + \frac{z-a}{2\pi i}\int \frac{f(t)\,dt}{(t-a)^2} + \cdots$$
$$+ \frac{(z-a)^n}{2\pi i}\int \frac{f(t)\,dt}{(t-a)^{n+1}} + \frac{1}{2\pi i}\int \frac{f(t)}{(t-z)}\left(\frac{z-a}{t-a}\right)^{n+1} dt.$$

But $f^{(s)}(a) = \dfrac{s!}{2\pi i}\displaystyle\int \frac{f(t)}{(t-a)^{s+1}}\,dt.$

and hence

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) + \cdots + \frac{(z-a)^n}{n!}f^{(n)}(a)$$
$$+ \frac{(z-a)^{n+1}}{2\pi i}\int \frac{f(t)}{t-z}\,\frac{dt}{(t-a)^{n+1}}$$

Denote the last term by L: let $t-a = r\,\varepsilon^{i\theta}$

Then
$$|L| = \frac{\rho^{n+1}}{2\pi}\left|\int_0^{2\pi}\frac{f(t)\,d\theta}{(t-z)(t-a)^n}\right| \qquad |\varepsilon^{i\theta}| = 1$$

But since $|z-a| = \rho$, $|t-a| = r$ $\therefore$ $|t-z| \geqq r-\rho$

Let M be the greatest Modulus of $f(t)$ along the circle. then
$$|L| < \rho^{n+1}\cdot\frac{1}{r^n(r-\rho)}M$$
$$< \left(\frac{\rho}{r}\right)^{n+1}\cdot\frac{M}{1-\frac{\rho}{r}}$$

The limit of this as $\underline{n}$ increases indefinitely is zero. Since $\rho < r$ Hence $\lim |L| = 0$.

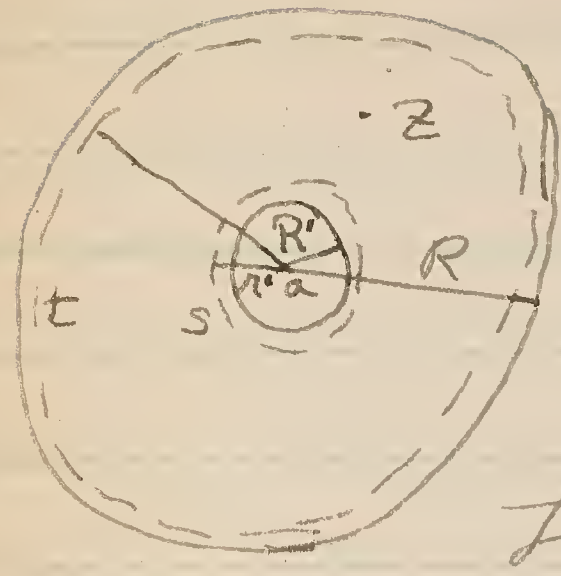
and the series is a converging series.

Such a series will often be denoted by
$$\mathfrak{P}(z-a) \qquad \text{(power series)}$$

A series which is convergent over the whole plane will be denoted by $\mathfrak{G}(z-a)$.

Ex. $\log(1+z)$; $\tan z$; ε^z; $\varepsilon^{\frac{1}{z-1}}$

<u>Laurent's Theorem</u> <u>A function which is holomorphic in a portion of the plane bounded by two concentric circles, with center $\underline{a}$ and finite radii, can be expanded in the form of a double series of integral powers, positive and negative, of $\underline{z-a}$, the series converging uniformly and unconditionally in the part of the plane between the circles.</u>



Let the radii be R, R' and let z be any point in the region bounded by circles r, r' such that $R > r > |z-a| > r' > R'$. Let t & s denote points on the circumferences r, r' resp.

Then

$$f(z) = \frac{1}{2\pi i}\int \frac{f(t)\,dt}{t-z} - \frac{1}{2\pi i}\int \frac{f(s)\,ds}{s-z} \qquad 27), 29)$$

But $\frac{1}{t-z} = \frac{1}{t-a+a-z} = \frac{1}{t-a}\cdot\frac{1}{1-\frac{z-a}{t-a}}$ $\quad z-a < t-a$ to make series converg.

$$= \frac{1}{t-a}\left[1 + \frac{z-a}{t-a} + \cdots + \frac{\left(\frac{z-a}{t-a}\right)^{n+1}}{1-\frac{z-a}{t-a}}\right]$$

Hence $\int \frac{f(t)\,dt}{t-z} = \int \frac{f(t)\,dt}{t-a} + (z-a)\int \frac{f(t)\,dt}{(t-a)^2} + \cdots$

$$+ (z-a)^n \int \frac{f(t)}{(t-a)^{n+1}}\,dt + \int \frac{f(t)}{(t-z)}\left(\frac{z-a}{t-a}\right)^{n+1} dt.$$

Now $\int \frac{f(t)\,dt}{(z-a)^{r+1}}$ is finite, since the expression to be integrated is finite along the circle path of integration.

Denote this integral by $2\pi i u_r$.

Hence

$$\frac{1}{2\pi i}\int \frac{f(t)\,dt}{t-z} = u_0 + (z-a)u_1 + (z-a)^2 u_2 + \cdots + (z-a)^n u_n$$
$$+ \frac{1}{2\pi i}\int \frac{f(t)}{t-z}\left(\frac{z-a}{t-a}\right)^{n+1} dt.$$

The modulus of the last term is less than $\frac{M}{1-\frac{\rho}{r}}\left(\frac{\rho}{r}\right)^{n+1}$ where $\rho = |z-a|$ and M is the greatest value of $|f(t)\,dt|$ on the circle of integration.

Since $\rho < r$, this quantity diminishes toward zero as n increases.

Hence the difference between $\frac{1}{2\pi i}\int \frac{f(t)\,dt}{t-z}$ and the first $n+1$ terms of the above series is a function whose modulus is diminishing toward zero as n increases to infinity. Hence the series is uniformly convergent. Further.

$$-\frac{1}{s-z} = -\frac{1}{s-a+a-z} = +\frac{1}{z-a}\cdot\frac{1}{1-\frac{s-a}{z-a}} \quad \left\{ \begin{array}{l} s-a < z-a \\ \text{series con.} \end{array} \right.$$

$$= \frac{1}{z-a}\left[1+\left(\frac{s-a}{z-a}\right)+\left(\frac{s-a}{z-a}\right)^2 + - - - \right]$$

$$\therefore -\frac{1}{2\pi i}\int\frac{f(s)}{s-z}ds = \frac{1}{z-a}\cdot\frac{1}{2\pi i}\int f(s)\,ds + - - - +$$

$$+\frac{1}{2\pi i}\frac{1}{(z-a)^{n+1}}\int (s-a)^n f(s)\,ds$$

$$+\frac{1}{2\pi i}\int\left(\frac{s-a}{z-a}\right)^{n+1}\frac{f(s)}{z-s}ds$$

But the modulus of the last term is less than $\dfrac{M'\left(\frac{r'}{\rho}\right)^{n+2}}{1-\frac{r'}{\rho}}$ where M' is the greatest value of $|f(s)|$ along the circle of radius r'. Since $\rho > r'$ it follows that the limit of the above expression is zero, as n increases to infinity.

Hence if $v_m = \frac{1}{2\pi i}\int (s-a)^{m-1} f(s)\,ds$.

then $-\frac{1}{2\pi i}\int\frac{f(s)\,ds}{s-z} = \frac{v_1}{z-a} + \frac{v_2}{(z-a)^2} + - - - -$

a series uniformly convergent.

Emil Leeb.

When a function is holomorphic over the entire plane outside a circle of center a it can be expanded in the form of a series of negative integral powers of $z-a$, the series converging everywhere in that part of the plane.

For in that case the term

$$\frac{1}{2\pi i}\int\frac{f(t)\,dt}{t-z} \text{ can be written}$$

$$\frac{1}{2\pi i}\int_0^{2\pi}\frac{dt}{\frac{t-z}{t-a}}f(t)$$ p. 53

Since by assumption the function $f(t)$ is not infinite at infinity, it either tends toward zero or to a finite limit K as r the radius $\doteq \infty$ of t-circle — also $\frac{t-z}{t-a}$ tends toward 1. Hence the above integral tends toward 0, or K.

Definitions of $\begin{cases}\text{Ordinary or regular point} \\ \text{Function said to behave regularly.}\end{cases}$

Domain of regular point. p 55

Accidental Singularity $\frac{1}{f(z)}$ regular

Essential Singularity The function has no definite value at the point either finite or infinite.

(nor does its reciprocal)

the uniform function must become infinite at an essential singularity for otherwise $f(z)(z-a)$ would be a uniform function at $z=a$. and could be expanded in the form
$A(z-a)+B(z-a)^2$ ------ and hence
$f(z) = A + B(z-a) + \cdots$
~~Further~~ $f(z)$ takes every value C at an essential singularity. For $\frac{1}{f(z)-C}$ has an essential singularity at $z=a$ and must therefore become infinite.
$\therefore f(z) - C$ must vanish.

Weierstrass's theorem

In the immediate vicinity of an essential singularity of a uniform function, there are positions at which the function differs from an assigned value C by a quantity not greater than a quantity ε as small as we please.

~~For let $|z-a| \neq \rho$~~

For values such that $0<|z-a|<\rho$ (Inside circle ρ = radius and not at a)
Consider the function $\frac{1}{f(z)-c}$ which may or may not have poles. If it have poles then $f(z)=c$ and the given value is actually attained. If it has no poles, then it is regular in the domain (everywhere) $0<|z-a|<\rho$ and can be expanded by Laurent's theorem in the form.

$$\frac{1}{f(z)-c} = S(z) + \frac{v_2}{z-a} + \frac{v_3}{(z-a)^2} + \cdots$$

$$= \mathfrak{S}(z) + \frac{T(z)}{z-a}$$

The series $S(z)$ converges everywhere on and within the circle, centre a and radius $\rho' < \rho$. The series $T(z)$ converges everywhere outside of a hence as $|S(z)|$ is finite for values of z within the circle ρ', and $|T(z)|$ is not zero. (why?) by choosing $|z-a|$ sufficiently small the right hand member of the preceding equation may have a modulus less than $\frac{1}{\varepsilon}$. Then

$$|f(z) - c| < \varepsilon. \quad \text{Q. E. D.}$$

<u>Analytic continuation of a function</u>

<u>Region of continuity</u>

<u>Element</u> of the function p 62-3

Aggregate of all the elements is called <u>Analytic function</u>

If the value of the function arrived at in the process of continuation from <u>a</u> to <u>z</u> is always the same the function is <u>uniform</u>. Otherwise <u>multiform</u>. The function is potentially given ~~by~~ an element. Some functions are identical with an element. e.g. $\sin z$, e^z

(That is first circle of convergence covers plane except centre a.)

Theorem I Uniform Functions

A function which is constant throughout any region of the plane, however small, or which is constant along any line however short, is constant throughout its region of continuity.

For if a is any point within such a region, then

$$f(z) = f(a) + f'(a)(z-a) + - - - -$$

But $f'(a) = \lim \frac{f(a+\delta a) - f(a)}{\delta a} = 0.$

and similarly all derivatives are zero

Same proof if $f(z)$ is constant along a line except that a & δa are taken along line.

Cor. I If two functions have the same value over any area or line in their common region of continuity then they have the same value in their common region of continuity

For the difference of the functions is 0. [2.8]

Cor II A function cannot be zero over any area or line of its region of continuity, however small, without being zero everywhere in that region.

Hence if a function exists which is evidently not zero everywhere, we conclude that its zeros are isolated points.

In any finite area of the region of continuity there can only be a finite number of zeros of a function if

no point of the boundary is an essential singularity of the function. (If a point is an essential singularity, it may have an infinite number of zeros infinitesimally near to it and therefore if an essential singularity lies on the boundary, there may be an infinite number of zeros in the region.) For if there were an infinite number of zeros, they would either form a continuous area or line, and the function would be zero, or they would cluster in infinite number about one or more points, such points are essential singularities, which is contrary to the hypothesis that the area considered is a part of the region of continuity of the function.

(Points where $f(z)$ assume same value are ~~in~~finite in number in infinite area.

Theorem II. <u>The multiplicity m of any zero a of a function is finite provided the zero be an ordinary point of the function which is not zero throughout its region of continuity, and the function can be expressed in the form</u>

$(z-a)^m \phi(z)$ <u>where $\phi(z)$ is holomorphic in the vicinity of a and a is not a zero of $\phi(z)$.</u>

for $f(z) = \frac{(z-a)^m}{\lfloor m} f^{(m)}(a) + \frac{(z-a)^{m+1}}{\lfloor m+1} f^{(m+1)}(a) - \text{———}$

$$= (z-a)^m \phi(z)$$

where $\phi(z)$ is evidently convergent.

Cor. I If infinity be a zero of order m of a function and an ordinary point, then the function can be expressed in the form

$$\frac{1}{z^m} \phi\left(\frac{1}{z}\right)$$

where $\phi\left(\frac{1}{z}\right)$ is continuous and not zero when $z = \infty$.

Cor. II The number of zeros of a function within a finite area of its region of continuity, a zero of multiplicity m being counted as m zeros, is finite if no essential point singularity of the function is on the boundary of this area

Theorem III A zero of a function is a zero of order $\left.\begin{matrix} m-1 \\ m+1 \end{matrix}\right\}$ of its derivative if $\begin{matrix} z = a \\ z = \infty \end{matrix}$ is a zero of order m of the function.

Cor. I If $f(z)$ is finite at ∞, then $z = \infty$ is a zero of order $\geqq$ **2** of the first derivative

Cor. 2. If $z = a$ is a zero of order n for $f(z)$ then

$$\frac{1}{2\pi i} \int \frac{f'(z)}{f(z)} dz = n$$

The integration being positive about a

Theorem IV A function must have an infinite value for some value, finite or infinite of the variable. For if M be a finite maximum modulus for f(z) for all points in the plane, then since

$$f^{(n)}(a) = \frac{\lfloor n}{2\pi i}\int \frac{f(z)\,dz}{(z-a)^{n+1}}$$ integrated about a

$|f^{(n)}(a)| < \frac{\lfloor n \, M}{r^n}$ where $r = |z-a|$. But by taking r sufficiently large we can make the right member as small as we please; hence $f^{(n)}(a) = 0$ since $f(z) = f(a) + f'(a)(z-a) + \text{- - - -}$ the function reduces to f(a) or is constant.

Cor I A function must have a zero value for some finite or infinite value of z for ~~if~~ the reciprocal must = ∞.

Cor II. A function must assume any given value at least once.

Cor. III Every function must have one singularity, either accidental or essential.

Theorem V A function which has a point c for an essential singularity can be expressed in the form

$$(z-c)^{-n}\,\phi(z)$$ p (23)

where n is a finite positive integer and $\phi(z)$ is a continuous function

in the vicinity of c.

Since $\phi(z)$ has an ordinary point at $c = z$

$$\phi(z) = u_0 + u_1(z-c) + \cdots u_{n-1}(z-c)^{n-1}$$
$$+ u_n(z-c)^n + \cdots$$
$$= u_0 + u_1(z-c) + \cdots + (z-c)^n Q(z-c).$$

where $Q(z-c)$ is a series of positive integral powers of $z-c$.

Hence $f(z) = \frac{u_0}{(z-c)^n} + \frac{u_1}{(z-c)^{n-1}} + \cdots \frac{u_{n-1}}{z-c} + \mathcal{L}(z-c)$

Cor. a function which has $z=\infty$ for an accidental singularity of multiplicity n can be expressed in the form.

$$z^n \phi\left(\tfrac{1}{z}\right).$$

Where $\phi\left(\frac{1}{z}\right)$ is a continuous function for very large values of $|z|$ and is not zero when $z = \infty$. It can also be expanded in the form.

$$a_0 z^n + a_1 z^{n-1} + \cdots a_{n-1} z + \mathcal{L}\left(\tfrac{1}{z}\right)$$

where $\mathcal{L}\left(\frac{1}{z}\right)$ is uniform, finite and continuous for very large values of z.

Criterion for distinction between accidental and essential singularities is: If $(z-c)^n f(z)$ n a finite integer is not infinite for $z = c$. then c is an accidental singularity.

Theorem VI. The poles of a function that lie in the finite part of a plane, are all the poles (of increased multiplicity) of the derivatives of the function that lie in the finite part of the plane

Note. A derivative function cannot have a simple pole in the finite part of the plane. (Ex. III p. 24.

Cor II. If c be a pole of $f(z)$ of multiplicity p we have, if $f(z) = (z-c)^{-p}\,\phi(z)$.

p 26 $$\frac{f'(z)}{f(z)} = -\frac{p}{z-c} + \frac{\phi'(z)}{\phi(z)}$$ where $\phi'(z)$ & $\phi(z)$ are holom.

Hence $\frac{1}{2\pi i}\int \frac{f'(z)}{f(z)}dz = -p$ the integral being taken positively around a circle so small as to exclude all other poles and zeros of $f(z)$.

Cor IV If a simple closed curve include a number n of zeros of a uniform function $f(z)$ and a number P of its poles, in both of which numbers account is taken of multiplicity, and if the curve has no essential singularities, then

$$\frac{1}{2\pi i}\int \frac{f'(z)}{f(z)} = n - P.$$ the integral being taken positively around the curve.

Ex. If $f(z) = a_0 z^n + a_1 z^{n-1} - - \; a_n$

then $f(z)$ has n roots.

For

over

$$\frac{1}{2\pi i}\int \frac{f'(z)\,dz}{f(z)} =$$

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{n a_0 \varepsilon^{in\theta} + (n-1)a_1 \frac{\varepsilon^{i(n-1)\theta}}{\rho} + (n-2)a_2 \frac{\varepsilon^{i(n-2)\theta}}{\rho^2} + \cdots}{a_0 \varepsilon^{in\theta} + a_1 \frac{\varepsilon^{i(n-1)\theta}}{\rho} + \frac{a_2 \varepsilon^{i(n-2)\theta}}{\rho^2} + \cdots}\,d\theta$$

where $z = \rho\,\varepsilon^{i\theta}$.

If ρ is made infinite this becomes

$$\frac{1}{2\pi}\int_0^{2\pi} n\,d\theta = n.$$

Ex. p. 76

Lect end.

~~Theo. VII If $z = \infty$ be a pole of $f(z)$ it is also a pole of $f'(z)$ only when it is a multiple root of $f(z)$~~ P. 36.

Theo. 8. A function which has no singularity in a finite part of the plane, and has $z = \infty$ for a pole, is a polynomial in z.

For.

$$f(z) = a_0 z^n + a_1 z^{n-1} + \cdots a_{n-1} z + \mathcal{L}\left(\tfrac{1}{z}\right).$$

where $\mathcal{L}$ is holomorphic in $\frac{1}{z}$ and is finite for $z = \infty$. As $f(z)$ has no singularities except at infinity, $\mathcal{L}\left(\frac{1}{z}\right)$ is never infinite and is constant. If n is finite (a position integer) the $f(z)$ is a rational integral function of z. If n is infinite $f(z)$ is a transcendental function.

Theo. 8. <u>A function all of whose singularities are accidental, is a rational meromorphic function</u>

Let the poles be $a_1, a_2 \dots a_\mu$ of orders $m_1, m_2, \dots m_\mu$ and let $z = \infty$ be a pole of order m.

Then

$$f(z) = a_m z^m + a_{m-1} z^{m-1} + \dots + a_1 z + f_0(z)$$

where $f_0(z)$ is not infinite when $z = \infty$

$f_0(z)$ is infinite at all poles of $f(z)$ except at $z = \infty$.

$$f_0(z) = \frac{b_{m_\mu}}{(z-a_\mu)^{m_\mu}} + \dots \frac{b_1}{z-a_\mu} + f_1(z)$$

$f_1(z)$ is not infinite for $z = a_\mu$ or $z = \infty$

Proceeding in this way.

$$f_\mu(z) = \frac{K_{m_1}}{(z-a_1)^{m_1}} + \dots + \frac{K_1}{z-a_1} + g(z)$$

where $g(z)$ is not infinite for any value of z and is hence a constant.

The combination of these functions gives $f(z)$ as a rational function.

Cor 1. <u>A function all the singularities of which are accidental, has as many zeros as it has accidental singularities in the plane.</u>

Cor 3. When a uniform function has no essential
p. 81. singularity, if the (finite) number of

its poles $c_1 \ldots c_m$ be m no one of them being at infinity and the number of its zeroes $a_1 \ldots a_m$ be m, no one of them being at ∞ then the function is

$$C \prod_{r=1}^{m} \left(\frac{z-a_r}{z-c_r}\right)$$ where c does not depend on z.

When $z = \infty$ is a pole of order p then the function has the form.

$$C \cdot \frac{\prod_{r=1}^{m} (z-a_r)}{\prod_{r=1}^{m-p} (z-c_r)}.$$

Multiform Functions. Algebraic Functions

A. Connected Surface.
A simply-connected surface
Multiply-connected surfaces.
Cross cuts. Loop cuts.
A surface is simply connected if it be resolved into two distinct pieces by every cross cut: but if there be any cross cut which does not resolve it into distinct pieces. the surface is multiply connected.
Multiply connected surfaces. Establishment of boundaries on such.
Surfaces not possessing a boundary.

Each of two distinct pieces, into which a simply connected surface S is resolved by a cross cut is simply connected.

Cor. 1 A simply connected surface is resolved by n cross-cuts into $n+1$ distinct pieces each simply connected. and an aggregate of m simply connected surfaces is resolved by n cross cuts into $m+n$ distinct simply connected surfaces.

II. If a resolution of a surface by m cross cuts into n distinct simply connected pieces be possible, and also a different resolution of the same surface by μ cross cuts into ν distinct simply connected pieces, then $m-n = \mu-\nu$.

Hence. If by any system of q cross cuts a multiply connected surface can be resolved into a number p of simply connected distinct pieces then $q-p$ is independent of the particular system of cross cuts and their configuration.

Surfaces classified according to number of cross cuts necessary to resolve it into a simply connected surface.

Connectivity.

n-ply connected if $n-1$ cross cuts

makes it simply connected. Then
$q = N-1$. $p = 1$. and since $q-p$ is invariant
$q-p = N-2$. or
$N = q-p+2$.
If a crosscut does not separate a surface into two distinct parts, the surface so cut has connectivity $N-1$. But if it divide the surface into two separate pieces of connectivity N_1, N_2 then $N_1+N_2 = N+1$.
IV. If s crosscuts be made in a surface of connectivity N. and divide it into $r+1$ separate pieces ($r \leqq s$.) of connectivities $N_1, N_2 \cdots N_{r+1}$
then $\sum N_i = N+2r-s$.

A loop cut is changed into a cross cut if from any point of it a cross-cut be made to any point of the boundary of the original surface. According as the loop cut does, or does not, divide the surface into distinct pieces, so also is the effect of the whole crosscut.
If a loop cut divide a surface of connectivity N into two surfaces of connectivity N_1, N_2 then
$N_1+N_2 = N+2$
The effect of making a simple hole in a continuous part of a surface is to increase its connectivity by one

The connectivity of a surface is not affected by a loop cut which does not divide the surface into distinct pieces.

If after any number of loop cuts be made in a surface of connectivity N there be $r+1$ distinct pieces of surface. of connectivity $N_1 \cdots N_{r+1}$ then $N_1 + \cdots\cdots N_{r+1} = N + 2r$.

Effect of a slit.

1. If the slip have no point on the boundary, then the connectivity is increased by 1.

2. If it has one extremity on the boundary the connectivity is unchanged.

A cross cut either increases by unity or diminishes by unity the number of distinct boundary lines of a surface. a loop cut increases by 2.

TB. The number of distinct boundary lines of a surface of connectivity N is $N - 2k$ where $k \geqq 0$.

Cor. A closed surface with a single boundary line is of odd connectivity since $m = 1$ $N = 2k+1$.

The surface is of class k.

~~(genus. If $N = 2p+1$. p = genus)~~

Cor. If the number of distinct boundary lines of a surface of connectivity N be N, any loop cut divides the surface into two distinct pieces.

L.

Theorem. If a closed surface of connectivity $2p+1$ (i.e. of genus p) be divided by circuits into any number of simply connected portions, each in the form of a curvilinear polygon, and if F be the number of polygons, E the number of edges, and S the number of angular points then

$$2p = 2 + E - F - S$$

P. 355 $\quad F + 2 = 2N + K + 2F - 2 - l$

Riemann Surfaces

The sheets of a Riemann Surface are joined along lines.

P 368. Branch lines.

I. A free end of a branch line in a surface is a branch point.

II. When a branch line extends beyond a branch point lying in its line, the sequence of the interchange of the sheets is not the same on the two sides of the point.

III. If two branch lines with different sequences of interchange have a common extremity, that point is either a branch point or an extremity of at least one other branch-line.

Page 183. line 8. should read "then unless it be a zero of $G_{n-1}(z)$ of an order at least equal to the order in which w' becomes infinite.

Page 194 line 15 should be

$$\frac{1}{F}\frac{\partial F}{\partial y} = \frac{1}{F_0}\frac{\partial F_0}{\partial y} - \frac{\partial}{\partial y}\sum_{\lambda=1}^{\infty}\frac{1}{\lambda}\frac{F_1^{\lambda}}{F_0^{\lambda}}$$

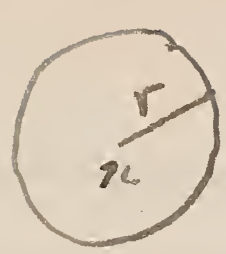
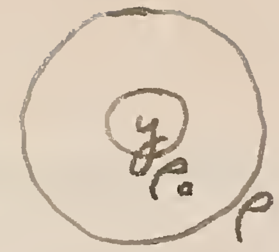
For derivation of this formula see next p.

Page 195 lines 7 & 8 from foot sub. of ξ, & η in last terms should be $\underline{s}$ instead of $\underline{l}$.

Page 79 line 9. $(z-a)^n\phi(z)$

~~Page~~ 89 line 7 from foot + should be −

Page 401 line 9. $\partial f/\partial w$. delete subscript 1.

Page 194   Line 7. "may be some limit of $|y|$". If y is small enough, it may be that the y's into the x's may be finite in lt.

$$(1) \qquad F = F_0\left(1 - \frac{F_1}{F_0}\right)$$

$$\frac{\partial F}{\partial y} = \frac{\partial F_0}{\partial y}\left(1 - \frac{F_1}{F_0}\right) - F_0\,\frac{F_0\frac{\partial F_1}{\partial y} - F_1\frac{\partial F_0}{\partial y}}{F_0^{\,2}}$$

$\div (1)$

$$\frac{1}{F}\frac{\partial F}{\partial y} = \frac{1}{F_0}\frac{\partial F_0}{\partial y}\frac{\cancel{\left(1-\frac{F_1}{F_0}\right)}}{\cancel{\left(1-\frac{F_1}{F_0}\right)}} - \frac{F_0\frac{\partial F_1}{\partial y} - F_1\frac{\partial F_0}{\partial y}}{F_0\left(F_0 - F_1\right)}.$$

$$= \frac{1}{F_0}\frac{\partial F_0}{\partial y} - \underline{A}$$

$$\frac{\partial}{\partial y}\sum_{\lambda=1}^{\infty}\frac{1}{\lambda}\frac{F_1^{\,\lambda}}{F_0^{\,\lambda}} = \frac{1}{1}\frac{\partial}{\partial y}\frac{F_1}{F_0} + \frac{1}{2}\frac{\partial}{\partial y}\frac{F_1^{\,2}}{F_0^{\,2}} + \frac{1}{3}\frac{\partial}{\partial y}\frac{F_1^{\,3}}{F_0^{\,3}} + \cdots.$$

$$= \frac{F_0\frac{\partial}{\partial y}F_1 - F_1\frac{\partial F_0}{\partial y}}{F_0^{\,2}} + \frac{F_0F_1\frac{\partial}{\partial y}F_1 - F_1^{\,2}F_0\frac{\partial}{\partial y}F_0}{F_0^{\,3}} + \cdots$$

Let $F_0 = a$, $F_1 = b$, $\frac{\partial}{\partial y}F_1 = m$, $\frac{\partial F_0}{\partial y} = n$

$$\underline{A} = \frac{am - bn}{a(a-b)} = \frac{m}{a} - \frac{bn}{a^2} + \frac{mb}{a^2} - \frac{b^2n}{a^3} + \frac{mb^2}{a^3} - \frac{b^3n}{a^4} + \cdots$$

$$= \frac{am - bn}{a^2} + \frac{abm - b^2n}{a^3} + \cdots$$

$$= \frac{\partial}{\partial y}\sum_{\lambda=1}^{\infty}\frac{1}{\lambda}\frac{F_1^{\,\lambda}}{F_0^{\,\lambda}}$$

$$\therefore \frac{1}{F}\frac{\partial F}{\partial y} = \frac{1}{F_0}\frac{\partial F_0}{\partial y} - \frac{\partial}{\partial y}\sum_{\lambda=1}^{\lambda=}\frac{F_1^{\,\lambda}}{F_0^{\,\lambda}}.$$

$$F_0 = By^m + cy^{m+1} + \cdots$$

$$\frac{\partial F_0}{\partial y} = mBy^{m-1} + \cdots$$

$$\frac{1}{F_0}\frac{\partial F_0}{\partial y} = \frac{m}{y} + G(y).$$

$$\frac{F_1^\lambda}{F_0^\lambda} = \sum_{\mu=0}^{\infty} G_{\lambda\mu}\, y^{-m\lambda+\mu}$$

Let $\lambda = 2$ to test.

$$F_1 = R(x) + y\,S(x) + y^2\,T(x) + \cdots$$

$$F_0 = B y^m + \cdots$$

$$F_1^2 = [R(x)]^2 + y^2[S(x)]^2 + 2\,R(x)\,S(x)\,y + \cdots$$

$$F_0^2 = B^2 y^{2m} + \cdots$$

$$\frac{F_1^2}{F_0^2} = \frac{[R(x)]^2}{B^2y^{2m}} + \frac{2R(x)\,S(x)\,y}{B^2y^{2m}} + \cdots$$

$$= G_{2,0}(x)\, y^{-2m} + G_{2,1}(x)\, y^{-2m+1} + \cdots$$

$\lambda = 2 \quad \mu = 0, \qquad \lambda = 2, \mu = 1 \quad \cdots$

(over)

38 line 8 $\frac{1}{m-1} = m^{-1} + m^{-2} + m^{-3} + \cdots$

P 195 $\frac{1}{F}\frac{\partial F(y\,\kappa)}{\partial y} = \sum_{l=1}^{s} \frac{1}{y-\eta_l} + P(y)$

$$\sum_{l=1}^{s} \frac{1}{y-\eta_l} = \frac{1}{y-\eta_1} + \frac{1}{y-\eta_2} + \frac{1}{y-\eta_3} + \cdots + \frac{1}{y-\eta_s} =$$

$$\frac{1}{\eta_1\left(\frac{y}{\eta_1}-1\right)} + \frac{1}{\eta_2\left(\frac{y}{\eta_2}-1\right)} + \cdots + \frac{1}{\eta_s\left(\frac{y}{\eta_s}-1\right)}$$

$$= \frac{1}{\eta_1}\left(\frac{y^{-1}}{\eta_1^{-1}} + \frac{y^{-2}}{\eta_1^{-2}} + \cdots\right) + \frac{1}{\eta_2}\left(\frac{y^{-1}}{\eta_2^{-1}} + \cdots\right) + \cdots$$

a) $= y^{-1} + \eta_1 y^{-2} + \eta_1^2 y^{-3} + \eta_1^3 y^{-4} + \cdots$

b) $+ y^{-1} + \eta_2 y^{-2} + \eta_2^2 y^{-3} + \eta_2^3 y^{-4} + \cdots$

$+ \cdots$

c) $+ y^{-1} + \eta_s y^{-2} + \eta_s^2 y^{-3} + \eta_s^3 y^{-4} + \cdots$

$$= \frac{s}{y} + (\eta_1 + \eta_2 + \cdots \eta_s) y^{-2} + (\eta_1^2 + \eta_2^2 + \cdots) y^{-3} + \cdots$$

$$= \frac{s}{y} + \sum_{\mu=1}^{\infty} S_\mu y^{-\mu-1}$$

~~Where~~ $S_\mu = \eta_1^\mu + \eta_2^\mu + \cdots \eta_s^\mu$.

$|y|$ must be greater than $|\eta|$ in order that series a) b) c) -- be convergent.

Line 16 In line 16 $\sum_{\mu=1}^{\infty} S_\mu y^{-\mu-1}$ the powers of y begin with the zero power. Breaking up the expression $-\frac{\partial}{\partial y}\sum_{\mu=-\infty}^{\infty} G_\mu y^\mu$ into two parts the second beginning with zero power of y we have

$$-\frac{\partial}{\partial y}\sum_{\mu=-\infty}^{\infty} G_\mu y^\mu = -\frac{\partial}{\partial y}\sum_{\mu=+\infty}^{-\infty} - \frac{\partial}{\partial y}\sum_{\mu=0}^{\infty} G_\mu y^\mu$$

Differentiating partially with respect to y and equating coefficients of like powers. After differentiating the exponent of y is decreased by 1 and $p-1=-\mu-1$ or $p=-\mu$

$$-\mu G_{-\mu} = S_\mu \quad \text{or} \quad S_\mu = \mu G_{-\mu}.$$

Ex. I page 4. $w = \frac{1}{z}$. (Recip. radii)

$$w = \frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2}$$

Let $x^2+y^2=\rho^2$ be eq. of a circle in z-plane

$$w = u+iv = \frac{x}{\rho^2} + i\frac{-y}{\rho^2} \quad \text{or} \quad \begin{cases} u = \frac{x}{\rho^2} \\ v = \frac{-y}{\rho^2} \end{cases} \quad \text{or} \quad \begin{cases} x = \rho^2 u \\ y = -\rho^2 v \end{cases}$$

$$x^2+y^2 = \rho^4u^2+\rho^4v^2 = \rho^2$$

or $u^2+v^2 = \frac{1}{\rho^2}$ or the eq. of a central circle in w-plane in which the radius is the recip. of the radius of the corresponding central circle in z-plane.

Making $\rho = 1$. $x^2+y^2=1$, $u^2+v^2=1$. or unit circle in z-plane corresponds to unit circle in w-plane. All points outside of unit circle in w-plane correspond to points inside unit circle in z-plane. Also $\rho=1$ $x=u$, $y=-v$ indicates that upper half of circle in w-plane corresponds to lower half of z plane and corresponding points are symmetrical with respect to u-axis.

Let $y = mx$ $\quad \begin{cases} u = \frac{x}{x^2+y^2} \\ v = \frac{-y}{x^2+y^2} \end{cases}$ $\quad$ $x^2u+y^2u = x$
$x^2v+y^2v=-y$
$xv+yu=0 \quad (a)$

From which $xv+mxu=0$, or $v=-mu$. or points on a line making an angle α with x-axis have corresponding points on a line making an angle $-\alpha$ with u-axis.

Let $x^2+y^2+ax+by+c=0$ be the eq. of any circle in z plane. (a) may be solved (owing to

its symmetry with respect to u & x; y & v.

$$\begin{cases} x = \dfrac{u}{u^2+v^2} \\ y = \dfrac{-v}{u^2+v^2} \end{cases} \quad (b)$$ on substituting

$$\left(\frac{u}{u^2+v^2}\right)^2 + \left(\frac{-v}{u^2+v^2}\right)^2 + a\frac{u}{u^2+v^2} - b\frac{v}{u^2+v^2} + c = 0$$

or $1 + au - bv + cu^2 + cv^2 = 0$ the corresponding figure is a circle in w-plane.

Ex. p. 4 $\quad 2z = (a-b)w + \dfrac{a+b}{w}$.

Let $u^2+v^2 = \rho^2$ be eq. of a central circle in w-plane. $\quad w = u+iv \quad z = x+iy$

$$2x + 2iy = (a-b)(u+iv) + \frac{(a+b)(u-iv)}{\rho^2}$$

$$2\rho^2 x = [\rho^2(a-b) + (a+b)]u$$

$$2\rho^2 y = [\rho^2(a-b) - (a+b)]v$$

$u^2+v^2 = \rho^2$ or

$$\text{a)}\quad \frac{x^2}{\left[\dfrac{\rho^2(a-b)+(a+b)}{2\rho}\right]^2} + \frac{y^2}{\left[\dfrac{\rho^2(a-b)-(a+b)}{2\rho}\right]^2} = 1 \text{ or } \frac{x^2}{m^2} + \frac{y^2}{n^2} = 1$$

$\sqrt{m^2 - n^2} = \sqrt{(a+b)(a-b)}$ or independent of ρ

$\therefore$ a) represents confocal ellipses

$$2x + 2iy = (a-b)(u+iv) + \frac{(a+b)(u-iv)}{u^2+v^2}$$

Let $v = mu$.

$$2x + 2iy = (a-b)(u+imu) + \frac{(a+b)(u-imu)}{u^2+m^2u^2}$$

$$2x = (a-b)u + \frac{(a+b)u}{u^2+m^2u^2}$$
$$2y = (a-b)mu - \frac{(a+b)mu}{u^2+m^2u^2}$$

$$mx + y = (a-b)mu$$
$$u = \frac{mx+y}{m(a-b)}$$

$$2x = \frac{mx+y}{m} + \frac{a+b}{(1+m^2)\frac{mx+y}{m(a-b)}}$$

$$2mx(mx+y) = (mx+y)^2 + \frac{m(a^2-b^2)}{1+m^2}$$

b) $$\frac{x^2}{\frac{a^2-b^2}{m(1+m^2)}} - \frac{y^2}{\frac{(a^2-b^2)m}{1+m^2}} = 1 \quad \text{or} \quad \frac{x^2}{A^2} - \frac{y^2}{B^2} = 1$$

$$\sqrt{A^2+B^2} = \sqrt{a^2-b^2}$$

Eq. b) confocal hyperbolas in z plane. also focal distance the same as in case of ellipses in eq a).

Ex p 4. $w = \log z = \log \rho \varepsilon^{i\theta}$
$= \log\rho + i\theta.$

$$u + iv = \log\rho + i\theta.$$
$$u = \log\rho. \qquad v = \theta.$$

See Harkness & Morley.

Sec. 9) $\dfrac{\partial v}{\partial y} = g(xy)$

$v = \int g(xy)\,dy = G(xy) + \varphi(x)$ Integrating relative to x leaves constant $\varphi(x)$

$v = \int \gamma(xy)\,dx = \Gamma(xy) + \Psi(y)$ Ditto y.

~~dv = Total Differential. top page 4. x∴ can be integrated.~~

Combining these values of V. $\varphi(x)$. & $\psi(y)$ can be determined.

11) Plot an example of a multiform function

$z = \rho_1 \varepsilon^{i\theta_1}$ $w = \log z$.

$w = \log z = \log \rho_1 + i(\theta_1 + 2K\pi)$

The part of w for each value of K is branch or the aggregate of one set of values of w.

12) Branch points. examples.

$w = \pm\sqrt{z}$. one $z=0$.

$w = \sqrt{z^2-1}$ two $z=1$ $z=-1$

$w = \sqrt[3]{z^3-1}$ three br. num. $z=1$, $\varepsilon^{\frac{2}{3}\pi i}$, $\varepsilon^{\frac{4}{3}\pi i}$

$w = \sqrt[3]{z}$ $z=0$. $w = \sqrt[3]{z}$, $\varepsilon^{\frac{2}{3}\pi i}\sqrt[3]{z}$, $\varepsilon^{\frac{4}{3}\pi i}\sqrt[3]{z}$

A function may be multiform for unrestricted variation but a branch may be uniform when the variable is restricted to particular regions in the plane.

13) A point in the plane, at which two or more ~~values~~ branches of the function assume the same value, and near which the branches are interchanged by appropriate modification of the path of z is called a branch point.

15) A root (or zero) of a function is the value of the variable for which the function vanishes. The infinities of a function are the points at which the value of the function is infinite.

16) A pole is an infinity such that in its immediate vicinity the reciprocal of the function is holomorphic.

18) If a is an infinity of $w = f(z)$ and the limit of $(z-a)^n w$ is finite a is a pole.

$$w = \frac{A_1}{z-a} + \frac{A_2}{(z-a)^2} + \cdots \frac{A_n}{(z-a)^n} + B_0 + B_1(z-a) + B_2(z-a)^2 + \cdots$$

Multiply by $(z-a)^n$ & pass to limit. $\lim (z-a)^n w = A_n$

n may be fractional or ...

$w = \frac{1}{\sqrt{z^2-1}}$ for $z = -1$, $z = +1$. $\frac{1}{2}$

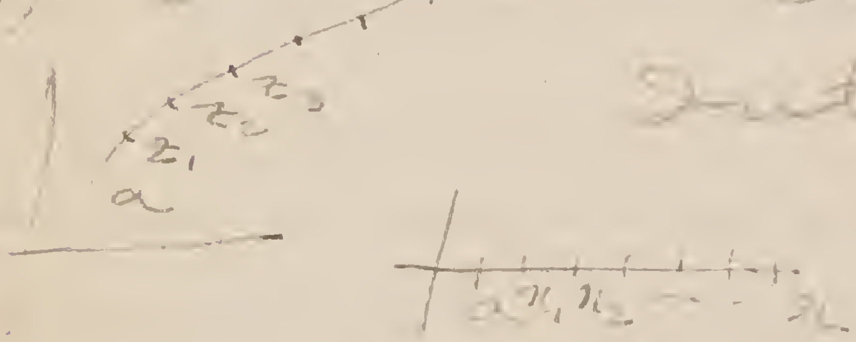
example of an infinite not a pole.

$w = \varepsilon^{\frac{1}{z}}$ for $z = 0$. $z^n \varepsilon^{\frac{1}{z}} \doteq \infty$ and therefore does not approach a finite limit.

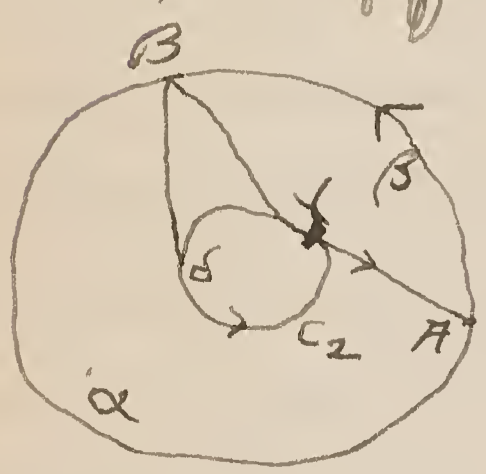
(A nice example of evaluation) Throw in the form $\frac{\varepsilon^{\frac{1}{z}}}{z^{-n}}$ &c. $= \infty$ ...

See H & M Ex. p. 182.

19) z z Same as definition of definite Integral in real variable.



Proof of Theorem §29 Consider the Int. taken pos. around C_1. We may replace the portion β of C_1 by a line AB tangent to C_2 at γ. (28) Instead of integrating from γ to B we may take a path along C_2 to δ and then to B and so by continually modifying the path in such a way that Cor. sec. 28 is clearly applicable we finally arrive at C_2 as the path of Integration.



Note on 31) $\frac{1}{z-a} \cdot \frac{1}{1-\frac{\delta a}{z-a}} = \frac{1}{z-a}\left[1 + \frac{\delta a}{z-a} + \frac{\frac{(\delta a)^2}{(z-a)^2}}{1-\frac{\delta a}{z-a}}\right] - \frac{1}{z-a}$

by Maclaurin's expansion.

§ 36) p 12 Ex. I $\underset{\rho=0}{lt}\int\frac{dz}{(a^2-z^2)^{\frac{1}{2}}} = 0$

$z-a=\rho\varepsilon^{i\theta}, \quad f(z)=\frac{1}{(a^2-z^2)^{\frac{1}{2}}}$

$(z-a)f(z)=\frac{z-a}{(a^2-z^2)^{\frac{1}{2}}}=\frac{\rho\varepsilon^{i\theta}}{\left[a^2-(a+\rho\varepsilon^{i\theta})^2\right]^{\frac{1}{2}}}$

$=\frac{\rho\varepsilon^{i\theta}}{\left[a^2-a^2-2a\rho\varepsilon^{i\theta}-\rho^2\varepsilon^{i2\theta}\right]^{\frac{1}{2}}}=\frac{\sqrt{\rho\varepsilon^{i\theta}}}{\left[-2a-\rho\varepsilon^{i\theta}\right]^{\frac{1}{2}}}$

$\underset{\rho=0}{lt}(z-a)f(z)=\frac{0}{(-2a)^{\frac{1}{2}}}=\underline{\underline{0}}$ ∴

Ex 2 $\lim\int\frac{dz}{(a-z)(z+a)^{\frac{1}{2}}}=\frac{\pi}{i}\left(\frac{z}{a}\right)^{\frac{1}{2}}$

$f(z)=\frac{1}{(a-z)(z+a)^{\frac{1}{2}}} \qquad f(z)(z-a)=\frac{-1}{(z+a)^{\frac{1}{2}}}=$

$z-a=r\varepsilon^{i\theta} \qquad \frac{-1}{\left[(r\varepsilon^{i\theta}+a)+a\right]^{\frac{1}{2}}}=\frac{-1}{(r\varepsilon^{i\theta}+2a)^{\frac{1}{2}}}$

$\underset{r=0}{lt.}\,f(z)(z-a)=-\frac{1}{(2a)^{\frac{1}{2}}} \qquad 2\pi i\cdot-\frac{1}{(2a)^{\frac{1}{2}}}=\underline{\underline{\frac{\pi}{i}\left(\frac{2}{a}\right)^{\frac{1}{2}}}}$

§ 38) Example $\lim\int\frac{dz}{\sqrt{1-z^n}}=\begin{matrix}2\pi & n=2\\ 0 & n>2\end{matrix}$

Circle, center at origin, $z=\rho\varepsilon^{i\theta}$

$\underset{\rho=\infty}{lt}\,z f(z)=\underset{\rho=\infty}{lt}\frac{\rho\varepsilon^{i\theta}}{\sqrt{1-(\rho\varepsilon^{i\theta})^n}}=\underset{\rho=\infty}{lt}\frac{1}{\sqrt{1-(\rho\varepsilon^{i\theta})^{n-2}}}=\begin{matrix}\frac{1}{i} & n=2\\ 0 & n>2\end{matrix}$

$\therefore \lim\int\frac{dz}{\sqrt{1-z^n}}=\begin{matrix}2\pi & n=2\\ 0 & n>2\end{matrix} \qquad \begin{matrix}2\pi i k=2\pi\\ 2\pi i k=0\end{matrix}$

§ 40) Ex 1. $L\int \frac{dz}{(a^2-z^2)^{\frac{1}{2}}} = 2\pi.$ enclosing $\underline{a}$, $-\underline{a}$

Circle, center origin and enclosing $a, -a$.

$z = a + \rho\varepsilon^{i\theta}$. $\qquad z f(z) = \dfrac{\rho\varepsilon^{i\theta} + a}{\sqrt{a^2 - 2a\rho\varepsilon^{i\theta} - \rho^2\varepsilon^{i2\theta} - a^2}}$

or $\underset{\rho=\infty}{L}\, z f(z) = \dfrac{1}{i} \qquad 2\pi i K = 2\pi.$

Ex. 2 40) $L\int \frac{dz}{\sqrt{(1-z^2)(c^2-z^2)}}$. curve including $\pm c, \pm 1$.

Circle, center origin including $\pm c, \pm 1$

$z = \rho\varepsilon^{i\theta} \pm (1-c)$

$$\underset{\rho=\infty}{L}\, z(fz) = \underset{\rho=\infty}{L} \frac{\rho\varepsilon^{i\theta} \pm (1-c)}{\sqrt{\left(1-\left[\rho\varepsilon^{i\theta} \pm (1-c)\right]^2\right)\left(c^2-\left[\rho\varepsilon^{i\theta} \pm (1-c)\right]^2\right)}} =$$

$$\underset{\rho=\infty}{L}\left[\frac{1 \pm \frac{1-c}{\rho\varepsilon^{i\theta}}}{\sqrt{\frac{c^2-(c^2+1)\left[\rho\varepsilon^{i\theta} \pm (1-c)\right]^2}{\rho^2\varepsilon^{i2\theta}} + \underset{\infty}{\underset{\parallel}{\frac{\rho^4\varepsilon^{i4\theta}}{\rho^2\varepsilon^{i2\theta}}}} + M}}\right] = \frac{1}{\infty} = 0$$

$\therefore \pi i K = 0$

Notes on Newton's Polygon p 183 F.

$$F_i(z) \equiv (z-a)^{-m_i}\left[c_1 + \delta_1(z-a) + \varepsilon_1(z-a)^2 + \cdots\right]$$

$$w = (z-a)^{-\lambda}\left[c + d(z-a) + \varepsilon(z-a)^2 + \cdots\right]$$

$$F(z) = (z-a)^{-\lambda n}\left[c + d(z-a) + \varepsilon(z-a)^2 + \cdots\right]^n +$$

$$(z-a)^{-\lambda(n+1)}(z-a)^{-m_1}\left[c + d(z-a) + \varepsilon(z-a)^2 + \cdots\right]^{n-1}\left[c_1 + d_1(z-a) + \cdots\right]$$

$$\cdots + (z-a)^{-\lambda}(z-a)^{-m_{n-1}}\left[c + d(z-a) + \cdots\right]\left[c_{n-1} + (z-a)d_{n-1} + \cdots\right]$$

$$+ (z-a)^{-m_n}\left[c_n + d_n(z-a) + \cdots\right] = 0.$$

since w satisfies. equation $F(z) = 0$.

$$c^n(z-a)^{-\lambda n} + c^{n-1}c_1(z-a)^{-m_1-\lambda(n+1)} + c^{n-2}c_2(z-a)^{-m_2-\lambda(n+2)}$$

$$+ c^{n-3}c_3(z-a)^{-m_3-\lambda(n+3)} + \cdots$$

$$c\, c_{n-1}(z-a)^{-m_{n-1}-\lambda} + c_n(z-a)^{-m_n} = 0. ?$$

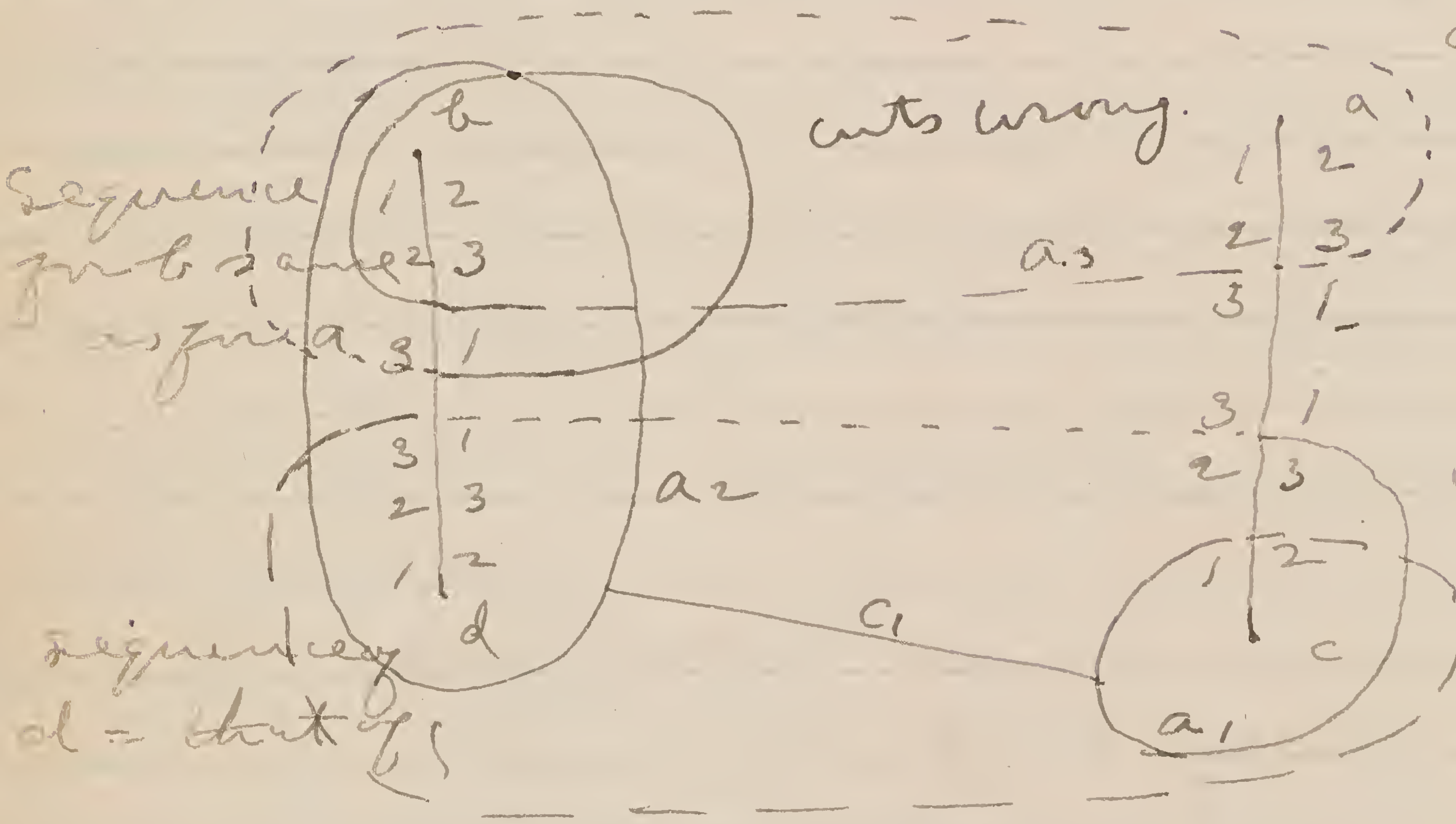
Exponents must be equal in pairs or triples, etc. since c's are not zero. Gives pairs for two points on line, etc.

If A_{n-r}, A_{n-s} are two points on line

$$(z-a)^{-m_{n-r}-\lambda r}\left(c_{n-r}c^r + \cdots \quad c^s_{n-s}\right) = 0.$$

As many terms in parenthesis as points on line.

$$w^3 = \frac{(z-a)(z-b)}{(z-c)(z-d)} \quad B.P.\ a, b, c, d.$$



Let $z - a = \rho\varepsilon^{i\theta}$

$w_1 = \rho^{\frac{1}{3}}(\varepsilon^{i\theta})^{\frac{1}{3}} R$

$w_2 = \rho^{\frac{1}{3}}(\varepsilon^{i2\pi\theta})^{\frac{1}{3}} R$

$= \rho^{\frac{1}{3}}\,\omega\, R$

$w_3 = \rho^{\frac{1}{3}}\omega^2 R$

Sequence for a = w_1, w_2, w_3, w_1

$z - c = \rho\varepsilon^{i\theta_1}$

$w_1 = \dfrac{1}{\rho^{\frac{1}{3}}(\varepsilon^{i\theta})^{\frac{1}{3}}} R'$

$w_3 = \dfrac{1}{\rho^{\frac{1}{3}}(\varepsilon^{i2\pi\theta})^{\frac{1}{3}}} R'$

$= \dfrac{1}{\rho^{\frac{1}{3}}\omega} R' = \dfrac{1}{\rho^{\frac{1}{3}}}\omega^2 R'$

$w_2 = \dfrac{1}{\rho^{\frac{1}{3}}}\omega R'$

Sequence w_1, w_3, w_2, w_1

$$\Omega = \Sigma(3-1) = 8$$

$n = 3 \quad \nu = 4$

$p = \dfrac{8-6+2}{2} = 2$

cuts $= 2p = 4$

second cut should have started in a boundary of a, and returned to other side of a,

$$f(w,z) = w^5 - (1-z^2)w^4 - \frac{4^4}{5^5} z^2(1-z^2)^4 = 0$$

$$\frac{\partial f}{\partial w} = 5w^4 - 4(1-z^2)w^3 = 0 \qquad w = \tfrac{4}{5}(1-z^2)$$

sub. $\frac{4^5}{5^5}(1-z^2)^5 - (1-z^2)^5\frac{4^4}{5^4} - \frac{4^4}{5^5}z^2(1-z^2)^4 = 0.$

$$\tfrac{4}{5}(1-z^2) - (1-z^2) - \tfrac{1}{5}z^2 = 0.$$

$$-1 - z^2 + z^2 = 0$$

$(z^2-1)^4 = 0$ $\qquad z = \infty$ twice

$z = +1, -1$ four times. w = infinity

(a) when $z = +1$, $w = 0$ five times.

$z = -1$ $\quad w = 0$ " " .

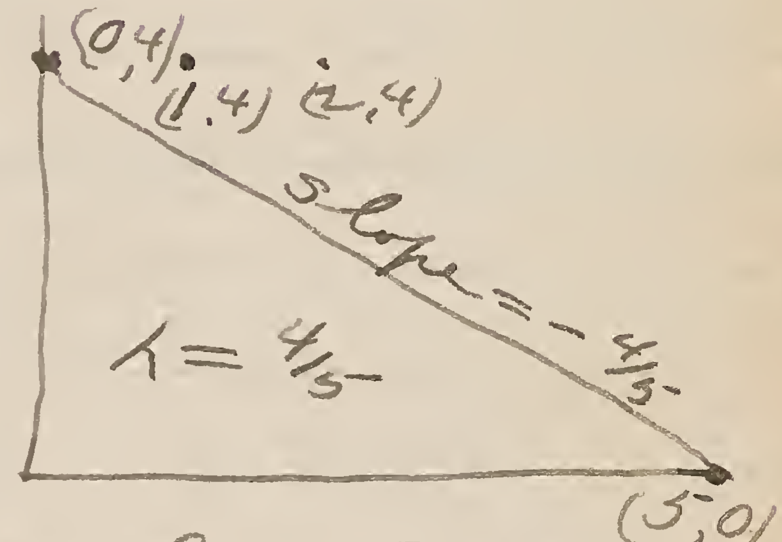
(a) $w = w' + 0$, $z = 1 + z'$

$$w'^5 - \left[1-(1+z')^2\right]w'^4 - \frac{4^4}{5^5}(1+z')^2\left[1-(1+z')^2\right]^4 = 0$$

$$w'^5 + 2zw^4 + z^2w^4 - \frac{4^4}{5^5}2^4z^4 + \frac{4^4}{5^5}24z^6 - \frac{4^4}{5^5}2^3z^7 - \frac{4^4}{5^5}9z^8 + \frac{4^4}{5^5}6z^9 - \frac{4^4}{5^5}z^{10} = 0$$

(b) $-\frac{4^4}{5^5}4^2z^4 + 2zw^4 + 2z^2w^4 + w^5 = 0.$

Coordinates are 0,4 1,4 2,4 5,0.



$\frac{p}{q} = \frac{4}{5}$ $\qquad z' = \rho^5$

$w' = v_1 z'^{4/5} = v_1\rho^4$

sub. in (b)

$$v_1^5\rho^{20} + 2v^4\rho^{21} + v^4\rho^{26} - \frac{4^6}{5^5}\rho^{20} = 0$$

Since powers must be equal in pairs

$v_1^5 = -\frac{4^6}{5^5}$ $\quad$ or $v_1 = \frac{4}{5}2^{2/5}\varepsilon_1$

where $\varepsilon_1 = \sqrt[5]{1}.$

$$w' = v_1 \rho^4 = \tfrac{4}{5}\, 2^{2/5} \varepsilon_1 \rho^4 + a\rho^5 + b\rho^6 + c\rho^7 + \cdots$$

$$z' = \rho^5 \qquad m = \frac{4^5}{5^5}$$

$$w'^5 + 2z'w'^4 + z'^2 w'^4 - m\, 2^4 z'^4 + m\, 24\, z'^6 - 8m z'^7 - 9m z'^8 + 6m z'^9 - m z'^{10} = 0.$$

$$w'^5 = \frac{4^5}{5^5} 2^2 \varepsilon_1^5 \rho^{20} + 5\frac{4^4}{5^4} 2^{8/5} \varepsilon_1^4 a \rho^{21} + 10\frac{4^3}{5^3} 2^{6/5}\varepsilon_1^3 \rho^{12} a^2 \rho^{10} + 10\frac{4^2}{5^2} 2^{4/5}\varepsilon_1^2 \rho^8 a^3 \rho^{15} + 5\cdot\tfrac{4}{5} 2^{2/5}\varepsilon_1 \rho^4 a^4 \rho^{20} + a^5\rho^{25} + 5\cdot\frac{4^4}{5^4} 2^{8/5}\varepsilon_1^4 \rho^{16} b\rho^6 + 5\cdot\frac{4^4}{5^4} 2^{8/5}\varepsilon_1^4 \rho^{16} c\rho^7 + 20\frac{4^3}{5^3} 2^{6/5}\varepsilon_1^3 \rho^{12} ab\rho^{11} + 20\frac{4^3}{5^3} 2^{6/5}\varepsilon_1^3 \rho^{12} bc\rho^{13} + 30\frac{4^2}{5^2} 2^{4/5}\varepsilon_1^2 \rho^8 a^2\rho^{10} b\rho^6 + 30\frac{4^2}{5^2} 2^{4/5}\varepsilon_1^2 \rho^8 a^2\rho^{10} c\rho^7 + a^4\rho^{20} b\rho^6 + a^4\rho^{20} c\rho^7 + 30\frac{4^3}{5^3} 2^{6/5}\varepsilon_1^3 \rho^{12} b^2\rho^{12} + \cdots$$

$$2z'w'^4 = 2\cdot 2\frac{4^4}{5^5} 2^{8/5}\varepsilon_1^4 \rho^{16}\cdot\rho^5 + 2\cdot 4\frac{4^3}{5^3} 2^{6/5}\varepsilon_1^3 \rho^{12} a\rho^5\cdot\rho^5 + 2\cdot 6\frac{4^2}{5^2} 2^{4/5}\varepsilon_1^2 \rho^8 a^2\rho^{10}\rho^5 + 2\cdot 4\cdot\tfrac{4}{5} 2^{2/5}\varepsilon_1 \rho^4 a^3\rho^{15}\rho^5 + 2a^4\rho^{25} + 2\cdot 4\frac{4^3}{5^3} 2^{6/5}\varepsilon_1^3 \rho^{12} b\rho^5\rho^5 + 2\cdot 12\frac{4^2}{5^2} 2^{4/5}\varepsilon_1^2 \rho^8$$

$$+z'^2 w'^4 = - - - - - - - - -$$

$$-\frac{4^5}{5^5} 2^4 \rho^{20} + \frac{4^5}{5^5} 24 \rho^{30} + - - - - - - -$$

Summing coefficients of like powers and placing them equal to zero in turn: we have.

ρ^{21} … $5\frac{4^4}{5^4} 2^{8/5}\varepsilon_1^4 a + 2\cdot 2\frac{4^4}{5^5} 2^{8/5}\varepsilon_1^4 = 0$

ρ^{22} … $10\frac{4^3}{5^3} 2^{6/5}\varepsilon_1^3 a^2 + \frac{4^4}{5^4} 2^{8/5}\varepsilon_1^4 b + 2\cdot 4\cdot\frac{4^3}{5^3} 2^{6/5}$

ρ^{23} … $10\cdot\frac{4^2}{5^2} 2^{4/5}\varepsilon_1^2 a^3 + 5\cdot\frac{4^4}{5^4} 2^{8/5}\varepsilon_1^4 c + 20\frac{4^3}{5^3} 2^{6/5}$

$$(a+b+c+d)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5 + 5a^4c + 5a^4d + 20a^3bc + 20a^3bd + 30a^2b^2c + 30a^2b^2d + 20ab^3c + 20ab^3d + b^4c + b^4d + 30a^3c^2 + \cdots$$

$$(a+b+c+d+\varepsilon)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 + 4a^3c + 12a^2bc + 12ab^2c + 4b^3c + 6a^2c^2 + 12abc^2 + 6b^2c^2 + 4ac^3 + 4bc^3 + c^4 + 4a^3d + 4a^3\varepsilon + 12a^2bd + 12a^2b\varepsilon + 12ab^2d + 12ab^2\varepsilon + 4b^3d + 4b^3\varepsilon + 12a^2cd + 12a^2c\varepsilon + 24abcd + 24abc\varepsilon + 12b^2cd + 12bc^2d + 12bc^2\varepsilon + 4c^3d + 4c^3\varepsilon + 6a^2d^2 + 6b^2d^2 + 6c^2d^2 + 12abd^2 + 12acd^2 + 12bcd^2 + 12a^2d\varepsilon + 12b^2d\varepsilon + 12c^2d\varepsilon + 24abd\varepsilon + 24acd\varepsilon + 24bcd\varepsilon + \cdots$$

$ab\zeta^{16} + \cdots$

$$w' = \frac{4}{5}2^{2/5}\varepsilon_1 z'^{4/5} - \frac{2^2}{5^2}z' + \frac{2^5(5-2^{2/5})}{5^3(2^{3/5}\varepsilon_1+2^3)}z'^{6/5} + \left(\varepsilon_1^3\frac{2^2}{5^4}2^{1/5} + 2^{-3/5}\frac{2^7}{5^4}\left(\frac{5-2^{3/5}}{2^{3/5}\varepsilon_1+2^3}\right)\varepsilon_1^4 - \frac{2\cdot 3}{5^3}2^{1/5}\varepsilon_1^3\right)z'^{7/6} + \cdots$$

$$a = -\frac{4}{25}$$

$$\varepsilon_1^3 a + 2\cdot 4\frac{4^3}{5^3}2^{6/5}\varepsilon_1^3 b = 0 \qquad b = \frac{2^5(5-2^{3/5})}{5^3(2^{3/5}\varepsilon_1+2^3)}$$

$$\varepsilon_1^3 ab + 2\cdot 6\frac{4^2}{5^2}2^{4/5}\varepsilon_1^2 a^2 = 0.$$

$$c = \varepsilon_1^3\frac{2^2}{5^4}2^{1/5} + 2^{-2/5}\cdot\frac{2^7}{5^4}\left(\frac{5-2^{2/5}}{2^{2/5}\varepsilon_1+2^3}\right)\varepsilon_1^4 - \frac{2\cdot 3}{5^3}2^{1/5}\varepsilon_1^3$$

$w^5 - (1-z^2)w^4 - \frac{4^4}{5^5} z^2 (1-z^2)^4 = 0$

$z \cdot z' = 1.$

$\cancel{w^5 - \left(\frac{1}{z'^2} - 1\right) w^4 - \frac{4^4}{5^5}\left(z' - \frac{1}{z'}\right)^4} = 0.$

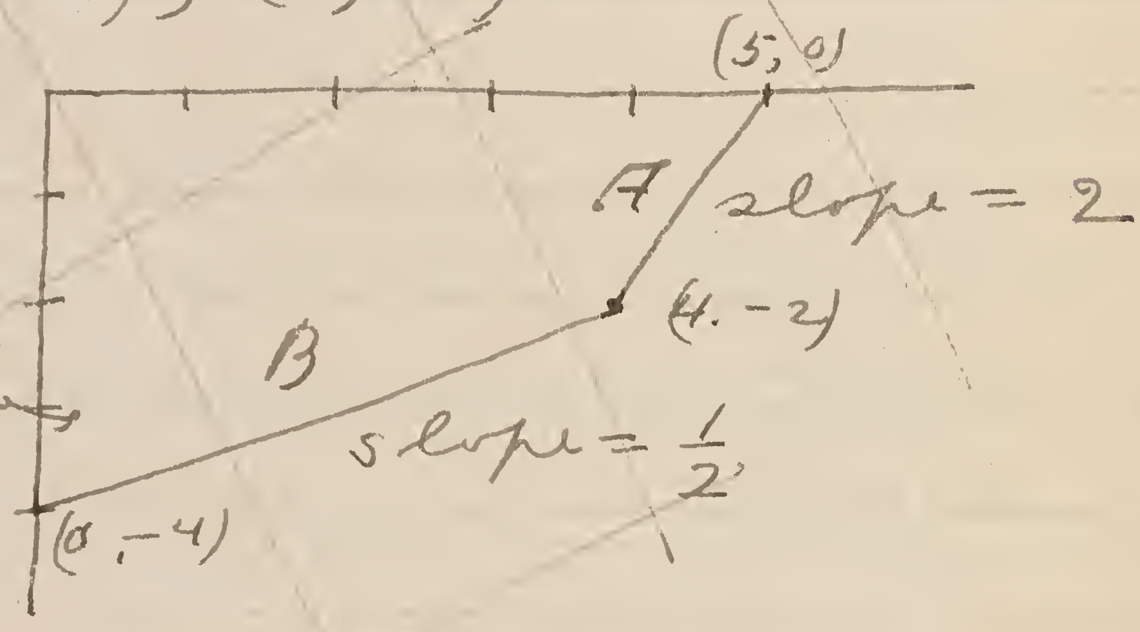
Drop primes.

$w^5 - z^{-2}w^4 + w^4 - \frac{4^4}{5^5} z^4 - \frac{4^5}{5^5}z^2 + \frac{4^4}{5^5}6 - \frac{4^5}{5^5}z^{-2} + \frac{4^4}{5^5}z^{-4}$

c) $w^5 + z^{-2}w^4 - \frac{4^4}{5^4} z^{-4} = 0$ Drop powers of z above lowest.

$z = 0$ is an infinity of c)

coordinates are $(5,0), (4,-2), (0,-4)$



A) $w' = z'^{-2}(y + ----)$

$z'^{-10}y^5 = +z'^{-10}y^4$

$y = +1., \; y = 0$ four times

$w' = z'^{-2}(+1 + ----$

B $w' = z'^{-\frac{1}{2}}(y + ---$

$z'^{-5/2} y^5 - z^2 z^{-2} y^4 + ------ + \frac{4^4}{5^5} z^{-4}$

$y^4 = \frac{4^4}{5^5}$

$y = \frac{4}{5}\sqrt[4]{\frac{1}{5}}\,\varepsilon_1 \qquad \varepsilon_1 = \sqrt[5]{1}.$

$w' = z'^{-2}\left(\frac{4}{5}\sqrt[4]{\frac{1}{5}}\,\varepsilon_1 + -----\right.$

$w^5 - (1-z^2)w^4 - \frac{4^4}{5^5} z^2(1-z^2)^4 = 0$

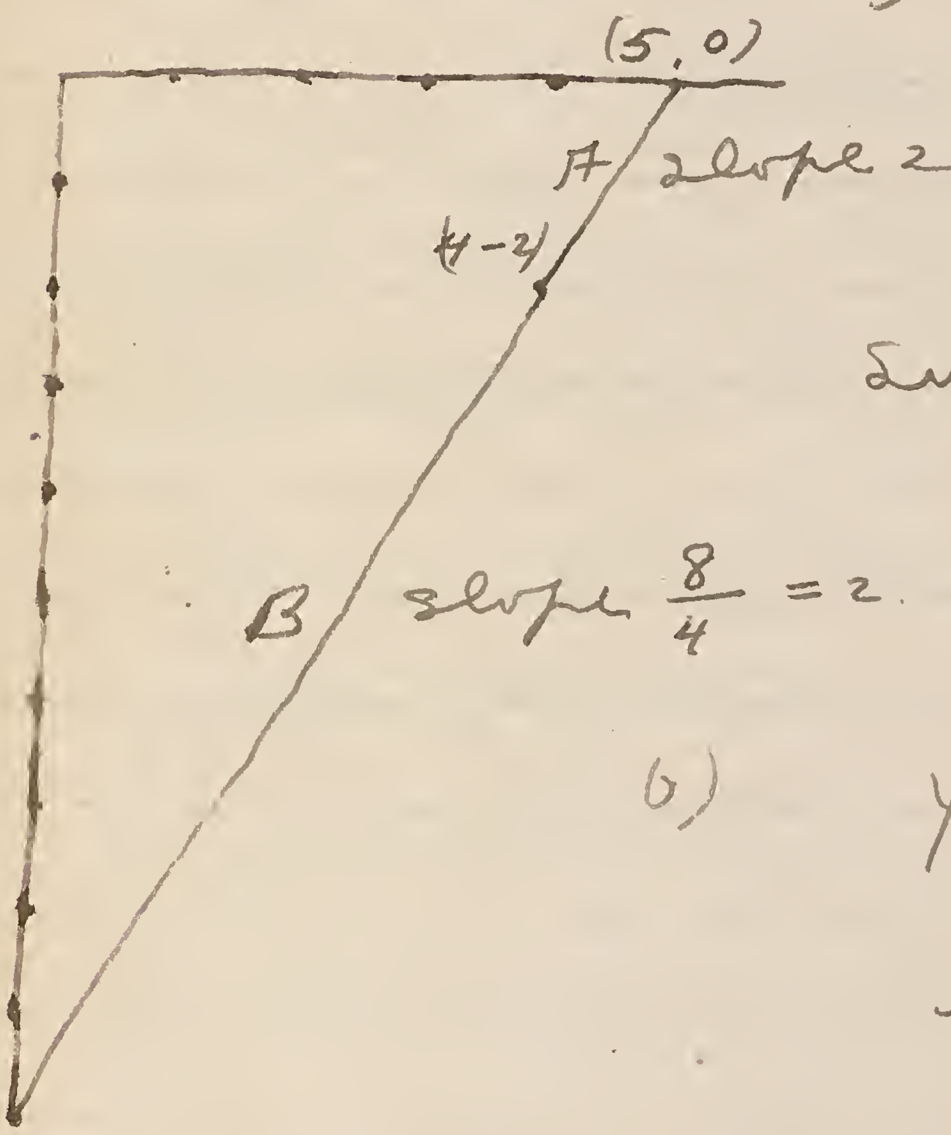
$zz' = 1$ Drop primes.

$w^5 - (1-\frac{1}{z^2})w^4 - \frac{4^4}{5^5}\frac{1}{z^2}(1-\frac{1}{z^2})^4 = 0.$

$w^5 - w^4 + w^4 z^{-2} - \frac{4^4}{5^5}(z^{-2} - 4z^{-4} + 6z^{-6} - 4z^{-8} + z^{-10}) = 0$

a) $w^5 + w^4 z^{-2} - \frac{4^4}{5^5} z^{-10} = 0.$ Coordinates are

(5,0) (4,-2) (0,-10)



$w' = z'^{-2}(\gamma + \beta z' + \cdots$

Sub in a)

$\gamma^5 z'^{-10} + 5\gamma^4 z'^{-9}\beta + \cdots + \gamma^4 z'^{-10} + 4\gamma^3 z'^{-9}\beta$

$+ \frac{4^5}{5^5} z'^{-8} - \frac{4^4}{5^5} z'^{-10}$

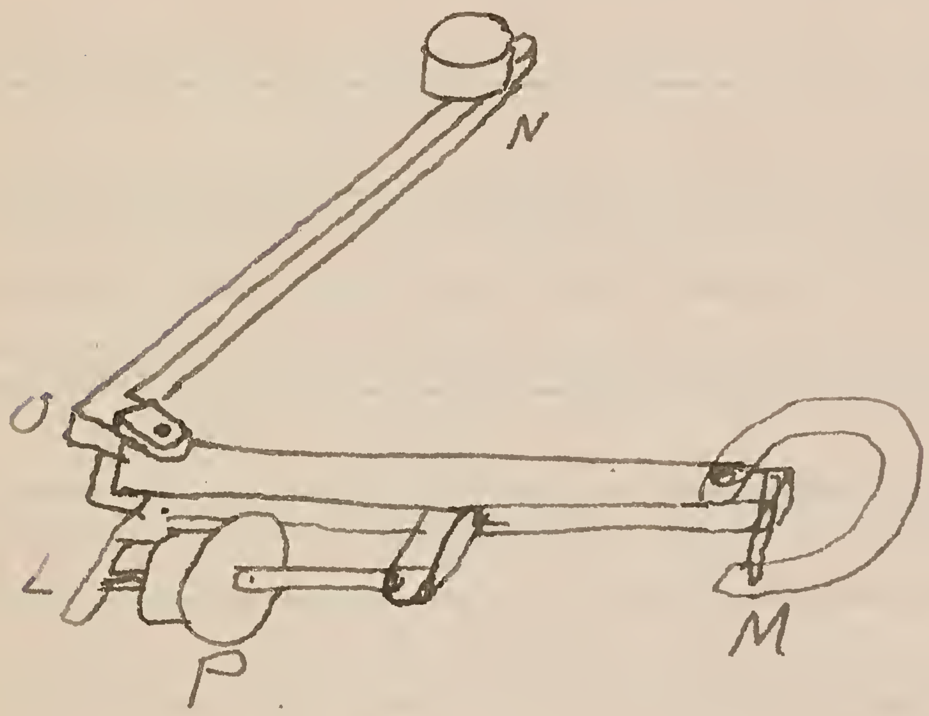
b) $\gamma^5 + \gamma^4 - \frac{4^4}{5^5} = 0.$

$5\gamma^4 + 4\gamma^3 = 0$

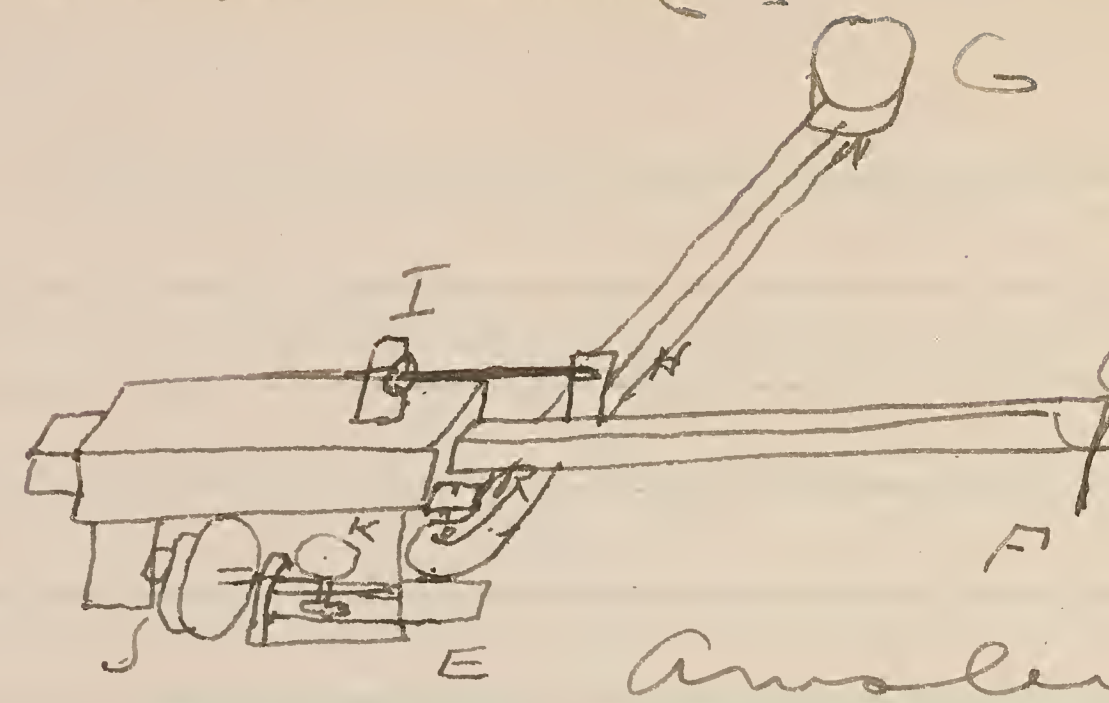
$\gamma + \frac{4}{5} = 0$ which is a factor of (a)

$\therefore (\gamma + \frac{4}{5})^2(\gamma^3 - \frac{3}{5}\gamma^2 + \frac{8}{5^2}\gamma - \frac{16}{5^3}) = 0$

 Calibration of Planimeter.



Amsler #2
Calkin's



Amsler.

Amsler's Planimeter consists of two arms EF & EG pivoted at E. Arm EG is free to rotate about E but it is held in place by a weight. The arm EF carries at one end a tracing point which is passed around the perimeter of the area to be measured. and this arm also carries the record wheel, or vernier, whose axis is in the same plane with it. The vernier is at J. and the number of whole revolutions is recorded at K. To adjust the arm. loosen the screw at ~~H~~ and slide arm along until desired unit mark comes opposite division mark at R. Then tighten R and a finer adjustment can be made by slow motion screw at I. This instrument was first tested to see if the axis of the vernier was parallel to arm E.F. which it should be to give accurate readings. The arms E.F. & E.G. were set at right angles to each other and the tracing

point was placed in a hole in a flat scale 1" from the fixed end which was used as a center of a 1" circle. The position of the scale was marked and the tracing point was revolved about the fixed center until the scale came around to its starting point. Then the vernier reading was recorded and the operation was repeated for five readings. Then the pivot point was placed outside the right angle and readings same as before were taken. If the results for both inside circle and outside circle are the same the axis of the second wheel is parallel to arm E.F. If the axis is not parallel the axis must be adjusted until the readings do come out equal. To determine the zero circle, an 8" circle was traced inside the right angle and readings recorded. and a 9" circle was traced outside. The zero circle as determined from the 8" circle is as follows. $\pi r^2 = \pi(8)^2 +$ vernier reading

$$r = \sqrt{\frac{\pi(8)^2 + 20.15}{\pi}} = 8.39''$$

From 9" circle $\qquad \pi r^2 = \pi(9)^2 -$ vernier r.

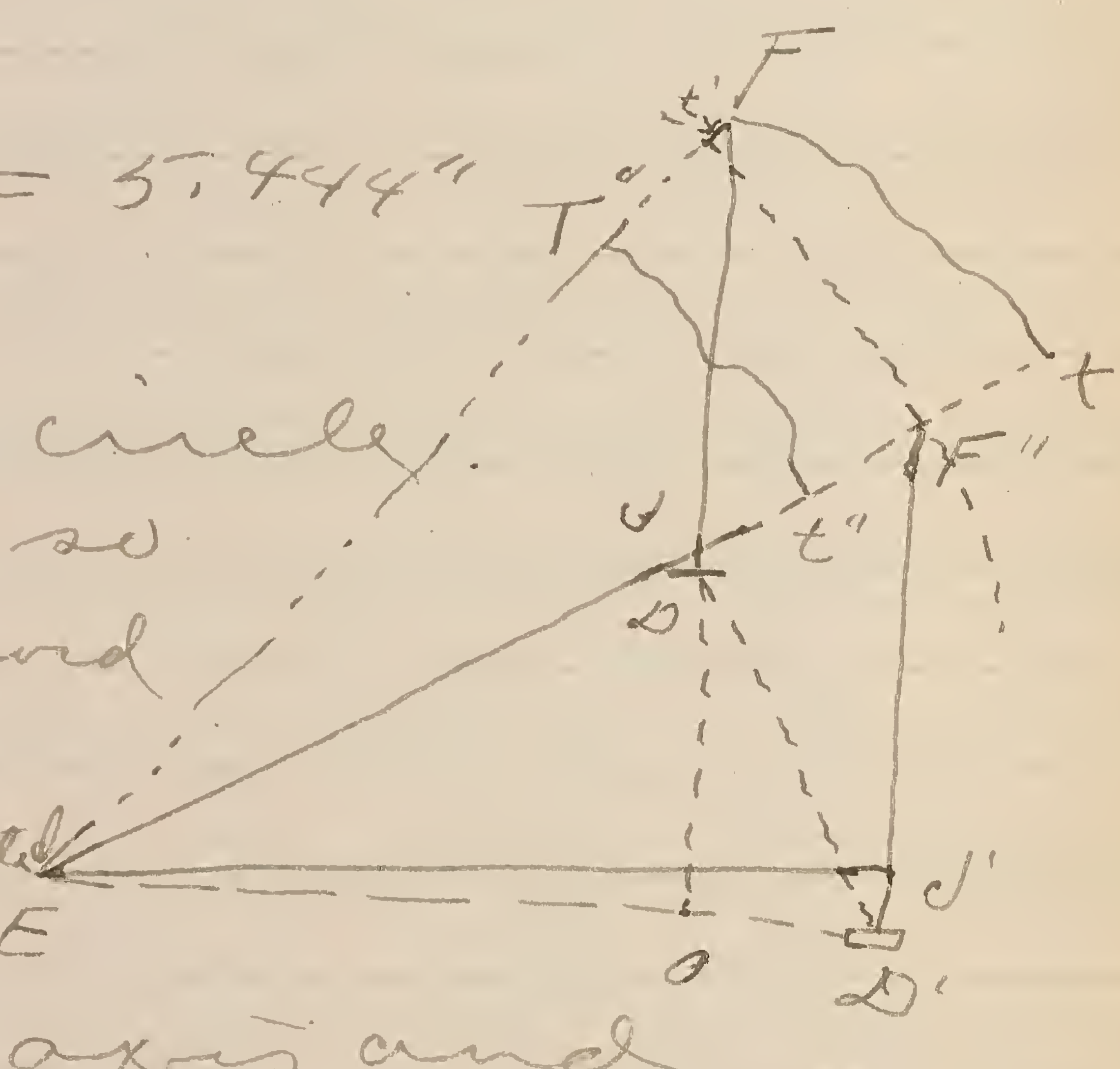
$$r = \sqrt{\frac{\pi(9)^2 - 34.38}{\pi}} = 8.37''$$

The area of a two inch circle was measured after the readings for the determination of zero circle was taken.
The Calkins exactly the same as #1. except zero circle was between a 5" and 7" circle. From 5"

$$r = \sqrt{\frac{\pi(5)^2 + 15.175}{\pi}} = 5.464''$$

From 7"

$$r = \sqrt{\frac{\pi(7)^2 - 60.628}{\pi}} = 5.444''$$



The theory of the zero circle.
If the arms be clamped so that the plane of the record wheel intersects the centre E. The graduated circle will travel in the direction of its axis and will not revolve. A circle generated around E is called the zero circle. If the instrument be unclamped and the tracing point be moved around an area in the direction of the hands of a clock outside the zero circle, the recording wheel will give a positive record, while if it be moved in the same direction inside the zero circle it will

give a negative record. In measuring areas that are very large and have to be measured by swinging the pointer around E as a center it is necessary to know the area of this zero circle which must be added to the determination given by the instrument; since for such areas that circumference is the initial point for measurement.

The movement of the record wheel.

From the preceding discussion it is seen that the record wheel does not register so long as its plane is radial, or so long as ESD $= 90°$. The amount of rotation due to variation in the angle ESD between the arms is, if an area be completely circumscribed, equal in opposite directions and hence does not affect the result, so that it is necessary to discuss merely the case of motion around the pole E with the angle ESD fixed. Suppose $\angle ESD$ constant and the tracing point to swing thro' an infinitesimal angle $F''EF' = d\theta$, the record wheel would move near

the path dD' more or less irregularly but subtending an equal angle DED'. The component of this motion which constitutes the record is $OD' = dR$ which is the projection of this path on a $\perp$ to dF. Since DED' is infinitesimal we have $DD' = DE\,d\theta$. also
$dR = OD' = DD' \cos ED'D$, but $ED'D = EDO$ from sim. Δ. Hence $dR = ED \cos EDO\,d\theta$.
Denote length of arm Ed by m. dF from pivot to tracing point by l. The distance dD from pivot to record wheel by n
$\angle EdD = B$. Let fall $\perp$ from E to Fd or FD produced at O. Then
$ED \cos EDO = OD = dO - dD = m \cos B - n$.
Hence $dR = (m \cos B - n)\,d\theta$ (1)
The infinitesimal area $F't\ F''t''$ lying adjacent to zero circle.
Let $EF' = r$. $EF'' = r'$ the radius of zero circle
let dA = area sought. $d\theta = FEt$.
Area $FEt = \frac{1}{2} r^2 d\theta$ and $F''Et' = \frac{1}{2} r'^2 d\theta$.
$dA = FEt - F''Et' = \frac{1}{2}(r^2 - r'^2)\,d\theta$. (2)
From ΔEdF, $r^2 = m^2 + l^2 + 2ml \cos B$ (3)
from right $\Delta ED'F''$, $r'^2 = m^2 + l^2 + 2nl$. (4)
subs. values of r^2 & r'^2 in (2) we have
$dA = l(m \cos B - n)\,d\theta$ -- (5) comparing (5). the differential equation for area. with (1) the corresponding eq. of record we see that $dA = l\,dR$ (6)

or integrating between limits 0 & R. since l is constant $A = lR$.
This equation shows that the area is equal to the length of arm from pivot to tracing point × by space registered on cir. of record wheel and is ~~in~~ independent of other dimensions of instrument.

Formula of Inst $A = lR$

	#	Inst Read 0 = 0	Diff in Inst. Read	Diff from mean ε	ε^2
1" ⊙ made	1	3.17	3.17	+.054	.002916
	2	6.30	3.13	+.014	.000196
	3	9.42	3.12	+.004	.000016
	4	12.51	3.09	−.026	.000676
	5	15.58	3.07	−.046	.002116
1" outside	1	3.08	3.08	+.016	.000256
	2	6.14	3.06	−.004	.000016
	3	9.20	3.06	−.004	.000016
	4	12.27	3.07	+.006	.000036
	5	15.32	3.05	−.014	.000196
2" ⊙	1	12.6	12.60	+.02	.0004
	2	25.19	12.59	+.01	.0001
	3	37.77	12.58	0	0
	4	50.33	12.56	−.02	.0004
	5	62.91	12.57	−.01	.0001

	Inst Read 0 = 0	Diff 20.13	ε^2
2" inside □ sq.	20.15	20.15	.0004
	40.22	20.07	.0036
	60.45	20.23	.01
	80.39	19.94	.0361
	100.65	20.26	.0169
		mean = 34.138	
2" outside	33.85	33.85	.083677
	68.84	34.99	.725906
	103.92	35.08	.886354
	135.02	33.10	1.077444
	168.69	33.67	.219024

calculated 2" cir. Area	12.5664
mean	12.58
Diff. m. & C	.0136
Error %	$0\frac{11}{100}$ %
mean error of 1 obs.	.0157
" " " result	.00705
Prob of 1 obs.	.01052
" " result	.00469

Area
Large circle from 8" ⊙ = 8.39" 9". 8.37"

Error in readings due to wear principally in inst.

$$f[(x+h)(y+k)] = f(h.k) + \left(x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\right) +$$

$$\frac{1}{1\cdot 2}\left(\frac{\partial f}{\partial x}x + \frac{\partial f}{\partial y}y\right)^2 + - \cdots$$